Modeling Quantum Computing - Matrix Mechanics

Purpose:

This paper was done to familiarize myself with some of the basic concepts of Quantum Computing. This work is based on models and examples from a number of books and articles on Quantum Computing. A list of some the books and articles used in this analysis are given at the end of this work. This modeling work was done using the Math Program Mathcad to Simulate various Quantum Mechanical and Quantum Computing Phenomena.
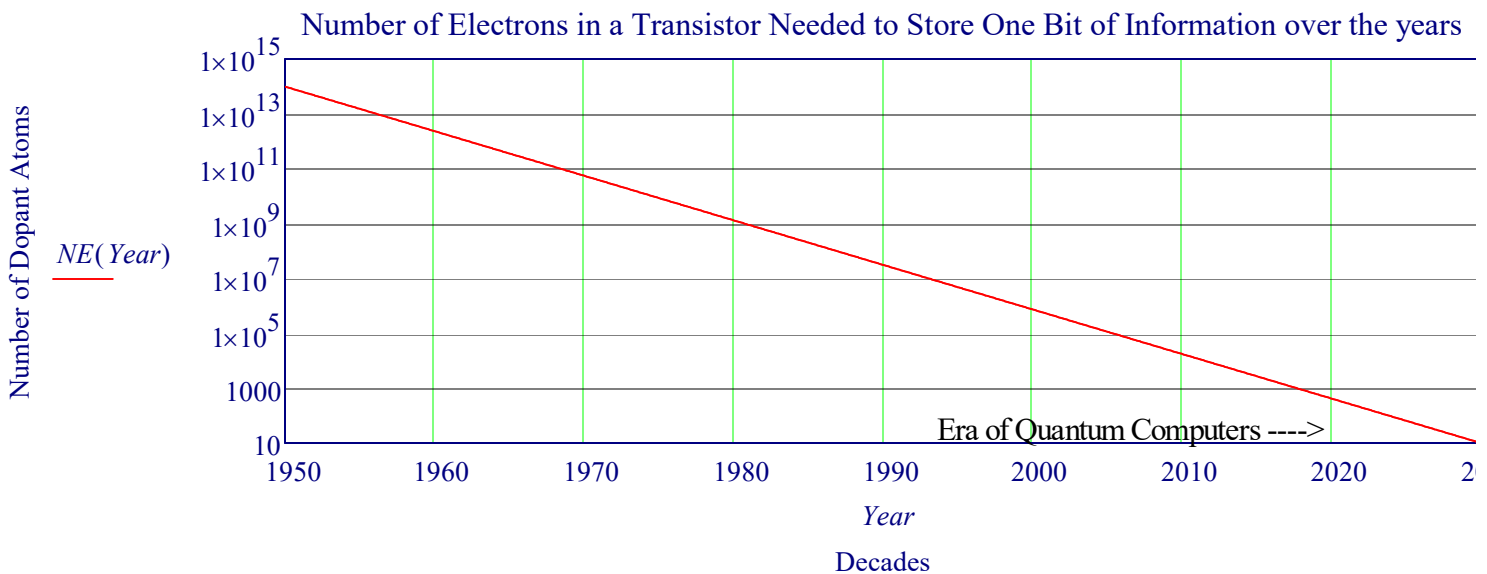
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I. Introduction

With the development of science and technology, leading to the advancement of civilization, new ways were discovered exploiting various physical resources such as materials, forces and energies. The history of computer development represents the culmination of years of technological advancements beginning with the early ideas of Charles Babbage and eventual creation of the first computer by German engineer Konard Zeise in 1941.

The number of atoms needed to represent a bit of memory has been decreasing exponentially since 1950. An observation by Gordon Moore in 1965 laid the foundations for what came to be known as "Moore's Law" – that computer processing power doubles every eighteen months. If Moore's Law is extrapolated naively to the future, it is learnt that sooner or later, each bit of information should be encoded by a physical system of subatomic size. The plot below shows the number of electrons required to store a single bit of information. An extrapolation of the plot suggests that we might be within the reach of atomic scale computations with in a decade or so at the atomic scale however. This is the point at which Moore's Law and the exponential growth classical computers comes to an end.

Decrease in the Number of Electrons Per Bit Per Year

$$NE(Y) := 10^{\left(14 - 13 \cdot \frac{Y - 1950}{80}\right)}$$

Number of Electrons in a Transistor Needed to Store One Bit of Information over the years

Number of Dopant Atoms

$NE(Year)$

Era of Quantum Computers ---->

Year

Decades

How does Qunatum Computing Work? An interview with Peter Shor, discoverer of the Shor Algorithm.

Peter Shor: "The key to factoring is identifying prime numbers, which are whole numbers divisible only by one and by themselves. (Five is prime. Six, which is divisible by two and by three, is not.) There are twenty-five prime numbers between one and a hundred, but as you count higher they become increasingly rare." Shor, drawing a series of compact formulas on the chalkboard, explained that certain sequences of numbers repeat periodically along the number line. The distances between these repetitions grow exponentially, however, making them difficult to calculate with a conventional computer.

"O.K., here is the heart of my discovery," he said. "Do you know what a diffraction grating is?" I confessed that I did not, and Shor's eyes grew wide with concern. He began drawing a simple sketch of a light beam hitting a filter and then diffracting into the colors of the rainbow, which he illustrated with colored chalk. "Each color of light has a wavelength," Shor said. "We're doing something similar. This thing is really a computational diffraction grating, so we're sorting out the different periods." Each color on the chalkboard represented a different grouping of numbers. A classical computer, looking at these groupings, would have to analyze them one at a time. A quantum computer could process the whole rainbow at once.

What is a Quantum Computer (QC)?

Quantum Computing: A type of computation whose operations can harness the phenomena of quantum mechanics, such as superposition, interference, entanglement, and teleportation. Devices that perform quantum computations are known as quantum computers.

Qubit: A qubit is a two-dimensional system that is in a state of 0 or 1 or both. Just as a classical bit has a state – either 0 or 1, a qubit also has a state. Two possible states for a qubit are the states |0> and |1>, which as you might guess correspond to the states 0 and 1 for a classical bit. Notation like ' |> ' is called the *Dirac notation*.

We can model it computationally:

$|\varphi> = \alpha |0> + \beta |1>$ where $|0> = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|1> = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\beta^2 = 1$, where α and β are complex numbers.

If a qubit gets measured, it will return classical bit value of 0 with probability of α^2 or a bit value of 1 with probability β^2.

Potential Applications: Cyber Security, Factorization, Breaking Codes (RSA), Simulate Quantum Phenomena. Rapid Prototyping and Testing of Chemical Reactions, Electronic and Material Properties, Molecular Folding, Calculating Fourier Transforms.

Key Concepts for QC: Quantum Behavior, Superposition, Entanglement, Interference, Teleportation, Computational Complexity/Scaling, and of the Physics of states in a quantum system.

Computational Complexity: Computational complexity studies the amount of time and space required to solve a computational problem. Another important computational resource is energy. Energy consumption in computation turns out to be deeply linked to the reversibility of the computation.

Reversibility: Consider a gate like the NAND gate, which takes as input two bits, and produces a single bit as output. This gate is intrinsically *irreversible* because, given the output of the gate, the input is not uniquely determined. The gate is an example of a reversible logic gate because, given the output of the gate, it is possible to infer what the input must have been. NAND Gate: A Boolean operator which gives the value zero if and only if all the operands have a value of one, and otherwise has a value of one (equivalent to NOT AND).

Erasure: Another way of understanding irreversibility is to think of it in terms of information erasure. If a logic gate is irreversible, then some of the information input to the gate is lost irretrievably when the gate operates – that is, some of the information has been erased by the gate. A computation is reversible if no information is erased during computation. Landauer's principle states that, in order to erase information, it is necessary to dissipate energy. A minimum of kBT ln2.

Types of QC Solution Models: Adiabatic, Topological, Quantum Annealing,

Entanglement is a different way of encoding information. If we have two particles that are entangled, the information about them is not encoded locally in each particle, but rather in the correlation of the two. This is the Principle of Non Locality.

Algorithms: Shor's (Factorization), Deutsch–Jozsa., Grover's (Search), Bernstein–Vazirani, Quantum phase estimation.

Measurements: Every measurable physical quantity, o, is described by a corresponding Hermitian operator, O, acting on the state ψ. Example: $H_{op}\psi = E\psi$ gives the Eigenvalue of Energy for state ψ. For every classically defined Function $F(x,p) \ni F_{op} = F(x, h/i \ d/dx)$ The Poisson Bracket Formulation for the momentum operator: $\dot{p}_k = \{p_k, H\}$

Obstacles: Decoherence, Error Correction, Scalability (need to scale to 1000 Bits for "practical" applications).

II. A Brief History of Quantum Computing

In 1981, Richard Feynman gave a lecture entitled "Simulating Physics with Computers" In this talk, he argued that a classical system could not adequately represent a quantum mechanical system.

In 1985, David Deutsch, a physicist at Oxford, suggested a more comprehensive framework for quantum computing in his 1985 paper. In this work, he describes in detail what a quantum algorithm would look like. He gave an algorithm that would run exponentially faster than any possible deterministic classical algorithm.

In 1993, Umesh Vazirani and his student Ethan Bernstein (BV) picked up where Deutsch and Jozsa left off. described an algorithm that showed clear quantum-classical separation even when small errors are allowed.

In 1994, Shor was a researcher in the mathematical division of Bell Labs in New Jersey. Shor studied the work of Deutsch, BV and Simon and realized he could construct an algorithm for factoring large numbers into two prime factors; factoring large numbers is believed to be intractable on a classical computer.

In 1999-2001, Yasunobu Nakamura built and demonstrated a functioning, controllable superconducting qubit. Nakamura used Josephson junctions to create a two-level system.

In 1995, Cirac and Zoller proposed an ion trap as the physical system to perform quantum computation.

1996 Shor and Robert Calderbank, and independently Andrew Steane, saw a way to finesse the seemingly show-stopping problems of quantum mechanics to develop quantum error correction techniques. Today, quantum error correction is arguably the most mature area of quantum information processing.

The Experimental Basis of Quantum Computing
Quantum Mechanics as a Model of Reality

Quantum mechanics or Matrix Mechanics is not quantum physics. Rather, it is the collection of mathematical tools used to analyze physical systems which are, to the best of anyone's ability to test, known to behave according to the laws of quantum physics.

III. A Few Key Concepts of Quantum Mechanics

A qubit is a quantum bit. We can represent a qubit as a two-dimensional complex Hilbert space, C2. The state of the qubit at any given time can be represented by a vector in this complex Hilbert space. A qubit system of say 100 qubits can handle 2^{100} states.

Relative phases of waveforms (states) are of fundamental importance for quantum algorithms in that they allow for constructive interference and destructive interference.

Qubits are abstract mathematical objects with certain specific properties.

Just as a classical bit has a state – either 0 or 1 – a qubit also has a state. Two possible states for a qubit are the states |0> and |1>, which as you might guess correspond to the states 0 and 1 for a classical bit. Notation like ' |>' is called the *Dirac notation*. The difference between bits and qubits is that a qubit can be in a state other than |0> or |1> It is also possible to form linear combinations of states, often called **superpositions**:

$$|\Psi> = \alpha\,|0> + \beta\,|0>$$

In the Dirac notation, the symbol identifying a vector is written inside a 'ket', and looks like $|a>$. We denote the dual vector for a (defined later) with a 'bra', written as $<a|$. Then **inner vector products** will be written as 'bra-kets' (e.g. $<a|b>$). Wile bras and kets are both elements of vector spaces, they are elements of different vector spaces. The ket corresponds to the normal vectors while the bra corresponds to a covector. Kets are part of one vector space while bras are part of the corresponding dual vector space.

The numbers a and ß are complex numbers, although for many purposes not much is lost by thinking of them as real numbers. Put another way, the state of a qubit is a *vector in a two-dimensional complex vector space*. The special states $|0>$ and $|1>$ are known as *computational basis states*, and form an *orthonormal basis* for this vector space. We can examine a bit to determine whether it is in the state 0 or 1. Rather remarkably, we *cannot examine a qubit to determine its quantum state*, that is, the values of a and ß. Instead, quantum mechanics tells us that we can *only acquire much more restricted information* about the quantum state. When we measure a qubit we get either the result 0, with probability $|a|^2$, or the result 1, with probability $|\beta|^2$. Naturally $|a|^2 + |\beta|^2 = 1$, since the probabilities must sum to one. Thus, in general a qubit's state is a *unit vector in a two-dimensional complex vector space*. This dichotomy between the unobservable state of a qubit and the observations we can make lies at the heart of quantum computation and quantum information. a qubit can exist in a continuum of states between |0> and |1>

– until it is observed. Let us emphasize again that when a qubit is measured, it only ever gives '0' or '1' as the measurement result – probabilistically. Because $|a|^2 + |\beta|^2 = 1$, we may write this as

$$|\psi\rangle = e^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right)$$

where θ, φ and γ are real numbers. We can ignore the factor of $e^{i\gamma}$ out the front, because it has no observable effects. The numbers θ and φ define a point on the unit three-dimensional sphere, as shown on page . This sphere is often called the Bloch sphere; it provides a useful means of visualizing the state of a single qubit. How much information is represented by a qubit? Paradoxically, there are an infinite number of points on the unit sphere.

Suppose we have two qubits. A two qubit system has four computational basis states denoted $|00>$, $|01>$, $|10>$, $|11>$. A pair of qubits can also exist in superpositions of these four states, so the quantum state of two qubits involves associating a complex coefficient – sometimes called an amplitude – with each computational basis state, such that the state vector describing the two qubits is

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$ An important 2 bit state is a Bell state: $\dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$

Characteristics of Quantum Computing
We denote the initial prepared state with a ket and label on the left of the wire |0>
We denote a single-qubit operator with a box containing the letter representing that operator straddling the line.
We denote a binary gate with an operator box spanning two quantum wires

IV. Unary Operators

Let us now cover the set of one-qubit, or unary, quantum operators. The first three operators we will examine are the Pauli matrices, X Y Z. These three matrices along with the identity matrix and all of their ±1 and ±i multiples constitute what is known as the Pauli group. A unary operator is a gate that takes single input bit, and a binary operator is one that takes two input bits. It is a linear transformation of the Hamiltonian Operator, H, that maps normalized (unit) vectors to other normalized vectors. Since H is 2-dimensional, a unary quantum operator can be represented by a 2×2 matrix.

X Operator: which is the NOT operator. It is denoted by the symbol $\oplus$

Y Operator, also denoted σ_y, which rotates the state vector about the y axis.

Z Operator, also denoted σ_z, which rotates the state vector about the z axis (also called the phase flip operator since it flips it by π radians or 180 degrees) (also known as the bit flip operator and can be referred to as x)

Rφ General phase shift operator. When we apply this operator we leave the state |0> as is and we take the state |1> and rotate it by the angle (or phase) denoted by φ, as specified in the matrix

S Operator, additional phase shift operators that are special cases of the Rφ matrix where $\varphi = \pi/2$. The S operator thus rotates the state about the z-axis by 90°.

$$R_\varphi := \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$$

T Operator which rotates the state about the z-axis by 45°. If we give φ the value of π=4. Note: $S = T^2$

H, Hadamard Operator: Note to TWK: Below matrices X and Y are named Xdot and Ydot. Define X and Y later

$$X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad R(\varphi) := \begin{pmatrix} 1 & 0 \\ 0 & e^{-\varphi} \end{pmatrix} \qquad T := \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}$$

$$S := \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad H := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad L0 := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad H \cdot L0 = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix}$$

Circuit diagrams:

$\alpha |0\rangle + \beta |1\rangle$ —— [X] —— $\beta |0\rangle + \alpha |1\rangle$

$\alpha |0\rangle + \beta |1\rangle$ —— [Z] —— $\alpha |0\rangle - \beta |1\rangle$

$\alpha |0\rangle + \beta |1\rangle$ —— [H] —— $\alpha \frac{|0\rangle + |1\rangle}{\sqrt{2}} + \beta \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

Below is the Dirac Notation for Block Sphere Equatorial +, -, i+, and i- States Shown on Following Page

$$|i+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \qquad\qquad |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|i-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) \qquad\qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

V. The Bloch Sphere

A Bloch sphere is a physical representation of all possible qubit states.
Each qubit is in its essence a vector on Bloch's sphere.

The Three ZXY Axes: State: |0> Z (Up),   State: (|+>) X Front (+),   State: (|i>) Y Side (i)

0 State (Z Up)	1 State (Z Down)	+ State (X Front)	- State (X Back)	i State (Y Right)	-i State (Y Left)

$$Z_u := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad Z_d := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad X_f := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad X_b := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad Y_r := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ i \end{pmatrix} \qquad Y_l := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Create a Bloch Sphere (XYZ) With Angles: θ, φ to Represent Any Qubit State

$$n := 50 \qquad i := 0..n \qquad j := 0..n \qquad \theta_i := \pi \cdot \frac{i}{n} \qquad \phi_j := 2\pi \cdot \frac{j}{n}$$

$$X_{i,j} := sin(\theta_i) \cdot cos(\phi_j) \qquad Y_{i,j} := sin(\theta_i) \cdot sin(\phi_j) \qquad Z_{i,j} := cos(\theta_i)$$
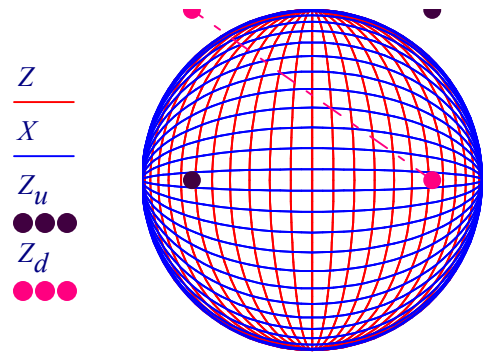
Next, the coordinates of a quantum qubit are calculated and displayed on the Bloch sphere as a white dot. As the polar and azmuthal angles are changed, you will need to rotate the figure to see where the white dot is on the surface of the Bloch sphere.
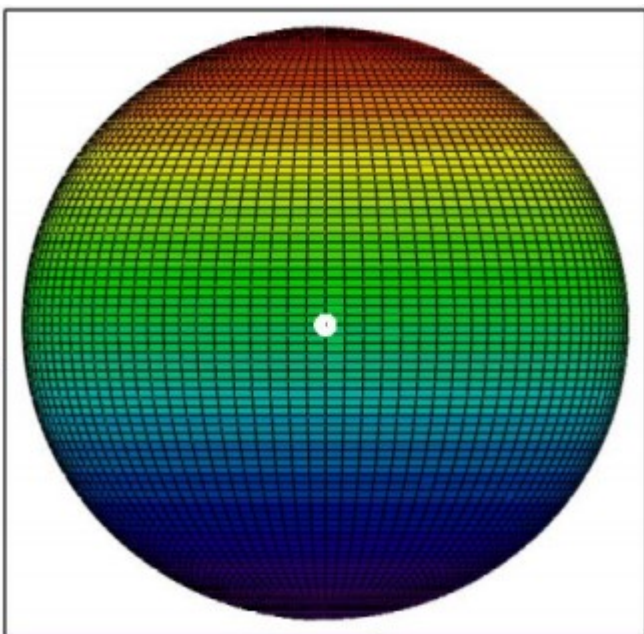
$$\theta1 := \frac{\pi}{2} \qquad \phi1 := 0$$

$$\Psi(\theta1, \phi1) := cos\left(\frac{\theta1}{2}\right) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + exp(\phi1) \cdot sin\left(\frac{\theta1}{2}\right) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Psi(\theta1, \phi1) = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix}$$

$$XX_{i,j} := sin(\theta1) \cdot cos(\phi1) \qquad ZZ_{i,j} := cos(\theta1) \qquad YY_{i,j} := sin(\theta1) \cdot sin(\phi1)$$

$$\frac{Z}{X} \\ \frac{Z_u}{\bullet\bullet\bullet} \\ \frac{Z_d}{\bullet\bullet\bullet}$$

Y, Z, X_b, X_b

$(X, Y, Z), (XX, YY, ZZ)$

$|0\rangle$

$|\Psi\rangle$

θ

$|i\rangle$

ϕ

$|+\rangle$

$|1\rangle$
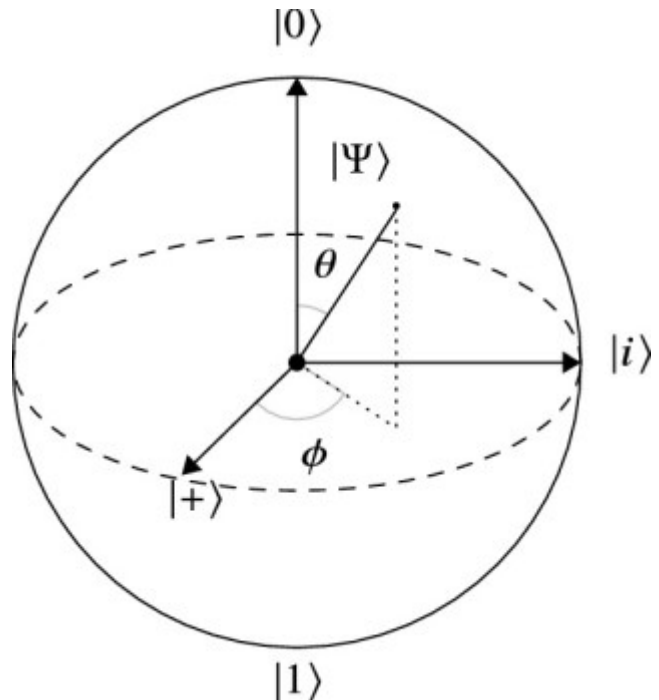
V. The Bloch Sphere - Continued

The Bloch sphere is a nice way to visualize quantum states and to identify orthogonal states. Furthermore, because diametrically opposite states in the Bloch sphere are orthogonal it also gives insight in which particular states are orthogonal. However, it is only possible to do this for single qubit states. When multiple qubit states are considered, t is not possible to visualize states in such a way

Dirac Notation for Block Sphere Equatorial +, -, i+, and i-

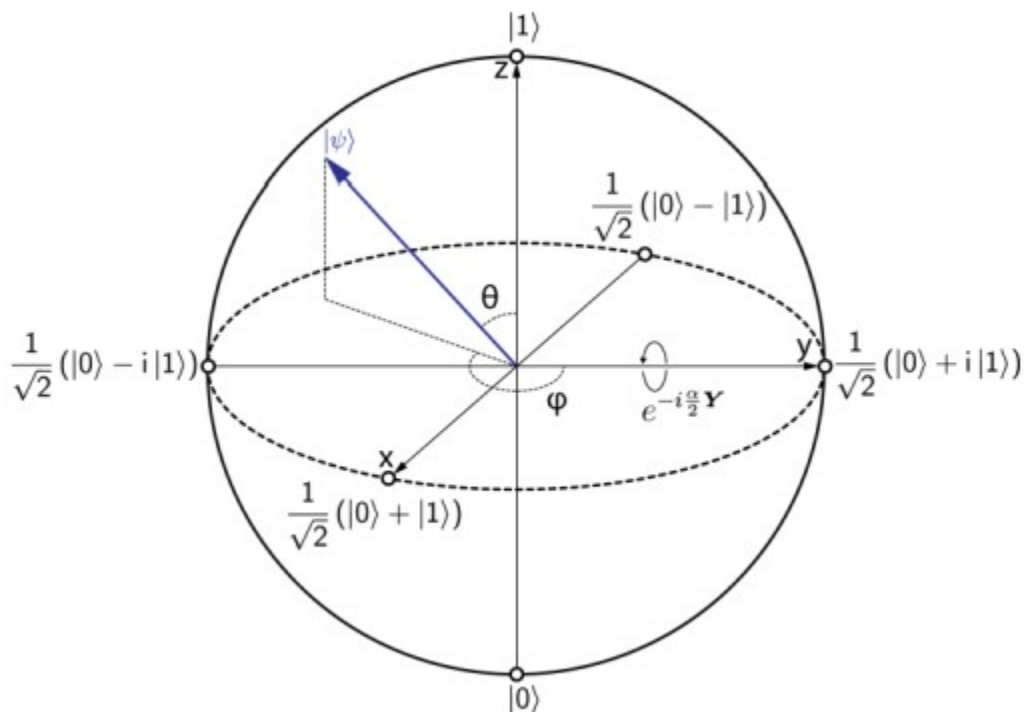$$|i+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$|i-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Note: This View Flips the "0" |0> |1> Orientation to "1" |1> |0> Orientation

The Clifford states in the Bloch sphere. *Programming for the quantum computer,* Dickel, 2016

VI. Matrix Mechanics of Spin States

See: *A Course in Quantum Computing*, Michael Loceff

Spin is a property that every electron possesses. Some properties like charge and mass are the same for all electrons, while others like position and momentum vary depending on the electron in question and the exact moment at which we measure. The spin, or more accurate term spin state of an electron has aspects of both. There is an overall magnitude associated with an electron's spin state that does not change. It is represented by the number 1/2, a value shared by all electrons at all times. But then each election can have its own unique vector orientation that varies from electron-to-electron or moment-to-moment.

Spin has a 3-D direction and a scalar magnitude and we can break it into the two aspects, its scalar magnitude,

$$S = |S| = \sqrt{S_x^2 + S_y^2 + S_z^2}$$

and a unit vector that embodies only its orientation (direction) $\dfrac{S}{|S|}$

S, the spin magnitude, is the same for all electrons under all conditions. Its value is $\dfrac{\sqrt{3}}{2}\hbar$ where $\hbar$ is a tiny number known as Plank's constant.

But, actual measurements of spin along the z-axis only give values of $S_z = \left(+\dfrac{\hbar}{2}\right)$ or $\left(-\dfrac{\hbar}{2}\right)$

The situation is shown on the below drawing.

$$0.577 \cdot \sqrt{\dfrac{3}{2}} = 0.707$$
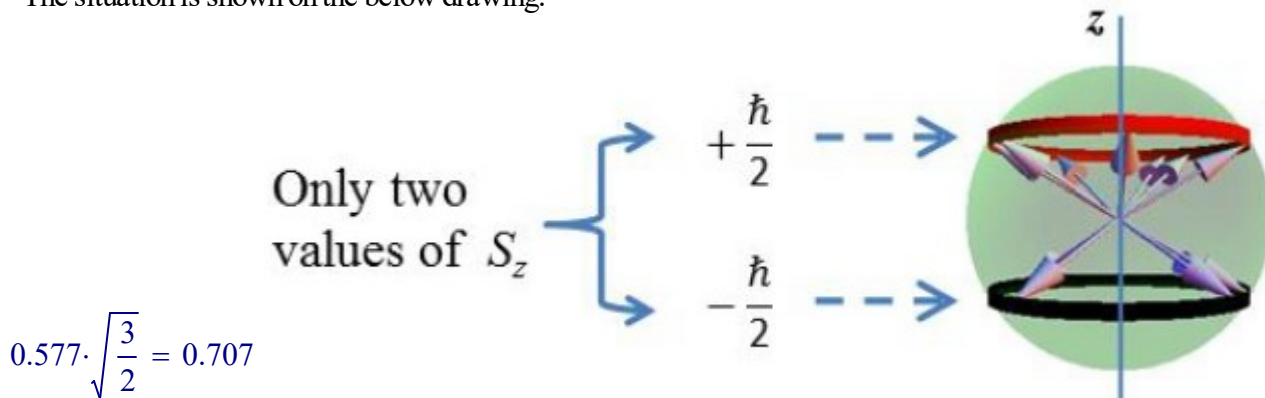
Why the is electron's projection onto the z-axis is not the entire length of the vector, that is, either straight up at $(+\sqrt{3}/2)\,\hbar$ or straight down at $(-\sqrt{3}/2)\,\hbar$. The electron stubbornly wants to give us only a fraction of that amount, $\approx 57.7\%$. This corresponds to two groups. The "up group" which forms the angle $\theta \approx 55°$.

This smaller value is due to the Heisenberg Uncertainty Principle. If the spin were to collapse to a state that was any closer to the vertical ±z-axis, we would have too much simultaneous knowledge about its x- and y-components (too close to (0) and its z-component (too close to ($\pm\sqrt{3}/2$) $\hbar$. This would violate the Heisenberg Uncertainty Principle, which requires the combined variation of these observables be larger than a fixed constant. Therefore, S z must give up some of its claim on the full spin magnitude, $\left(\dfrac{\sqrt{3}}{2}\right)\hbar$

|+i> and |-i> States. We give a name to the state of the electrons in the (+) group: we call it the $|+\rangle_z$ state (or simply the $|+\rangle$ state, since we consider the z-axis to be the preferred axis in which to project the spin).

We say that the (-) group is in the $|-\rangle_z$ (or just the $|-\rangle$ state. Verbally, these two states are pronounced "plus ket" and "minus ket.

VI. Matrix Mechanics of Spin States - Continued

The basic principles of quantum theory can be demonstrated very simply by exploring the properties of electron spin using Heisenberg's formulation of quantum mechanics which is usually referred to as matrix mechanics. The matrix formulation provides clear illustrations of the following essential quantum mechanical concepts: eigenvector, operator, eigenvalue, expectation value, the linear superposition, and the commutation relations.

Four quantum numbers are required to describe an electron in quantum mechanics. The last of these is the spin quantum number, s. The electron has a spin component in the x-, y-, and z-directions and for each of these directions the electron can have a value of ↑ spin-up or ↓ spin-down, or +1 and -1 in units of h/4π. These six spin states are represented by vectors as is shown below.

<div align="center">

X Y Z Axes Spin States (Up and Down)   **Transpose of Complex Conjugate (Shift")**^{T}

</div>

$$S_{xu} := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad S_{xd} := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \left(\overline{S_{xu}}\right)^T = (0.707 \quad 0.707) \qquad \left(\overline{S_{xd}}\right)^T = (0.707 \quad -0.707)$$

$$S_{yu} := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ i \end{pmatrix} \qquad S_{yd} := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad \left(\overline{S_{yu}}\right)^T = (0.707 \quad -0.707i) \qquad \left(\overline{S_{yd}}\right)^T = (0.707 \quad 0.707i)$$

$$S_{zu} := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad S_{zd} := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \left(\overline{S_{zu}}\right)^T = (1 \quad 0) \qquad \left(\overline{S_{zd}}\right)^T = (0 \quad 1)$$

Let's look at the the **y-direction spin states because they are complex**, and therefore are slightly moredifficult to deal with. In Dirac notation these four bra and ket Y vectors are written as $|S_{yu}\rangle, |S_{yd}\rangle$, $\langle S_{yu}|$, and $\langle S_{yd}|$. Note that the **bra-vectors are the transpose of the complex conjugate of the ket-vectors**. It is also easy to show that these spin vectors in the x-, y-, and z-directions form orthonormal basis sets. That means they are normalized and orthogonal to each other.

$$\left(\overline{S_{xu}}\right)^T \cdot S_{xu} = 1 \qquad \left(\overline{S_{xd}}\right)^T \cdot S_{xd} = 1 \qquad \left(\overline{S_{xu}}\right)^T \cdot S_{xd} = 0$$

$$\left(\overline{S_{yu}}\right)^T \cdot S_{yu} = 1 \qquad \left(\overline{S_{yd}}\right)^T \cdot S_{yd} = 1 \qquad \left(\overline{S_{yu}}\right)^T \cdot S_{yd} = 0$$

$$\left(\overline{S_{zu}}\right)^T \cdot S_{zu} = 1 \qquad \left(\overline{S_{zd}}\right)^T \cdot S_{zd} = 1 \qquad \left(\overline{S_{zu}}\right)^T \cdot S_{zd} = 0$$

In Dirac notation we would write the first row as: $\langle S_{xu}|S_{xu}\rangle = \langle S_{xd}|S_{xd}\rangle = 1$, $\langle S_{xu}|S_{xd}\rangle = 0$. In other words the **projection of the spin states onto themselves is 1** (normalized) and the projection onto the other state is zero (orthogonal). Momentum is only in one direction. Spin can only be ↑ or ↓.

The calculations above for the y-direction spin vectors are shown explicitly below.

$$(0.707 \quad -.707i) \cdot \begin{pmatrix} .707 \\ .707i \end{pmatrix} = 1 \qquad (0.707 \quad .707i) \cdot \begin{pmatrix} .707 \\ -.707i \end{pmatrix} = 1 \qquad (0.707 \quad -.707i) \cdot \begin{pmatrix} .707 \\ -.707i \end{pmatrix} = 0$$

It is easy to show that **x- and z-spin states are not orthogonal** to one another. Any two different spin directions are not orthogonal. $\langle S_{xu}|S_{zu}\rangle = 0.707$, for example. **This is a 45° angle.**

$$\left(\overline{S_{xu}}\right)^T \cdot S_{zu} = 0.707 \quad \left(\overline{S_{xu}}\right)^T \cdot S_{zd} = 0.707 \quad \left(\overline{S_{xd}}\right)^T \cdot S_{zu} = 0.707 \quad \left(\overline{S_{xd}}\right)^T \cdot S_{zd} = -0.707$$

This of course means that, for example, $|S_{xu}\rangle$ and $|S_{xd}\rangle$ can be written as linear superpositions of $|S_{zu}\rangle$ and $|S_{zd}\rangle$, and $|S_{zu}\rangle$ and $|S_{zd}\rangle$ can be written as linear superpositions of $|S_{xu}\rangle$ and $|S_{xd}\rangle$.

$$S_{xu} = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix} \quad \frac{1}{\sqrt{2}} \cdot S_{zu} + \frac{1}{\sqrt{2}} \cdot S_{zd} = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix} \quad S_{xd} = \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix} \quad \frac{1}{\sqrt{2}} \cdot S_{zu} - \frac{1}{\sqrt{2}} \cdot S_{zd} = \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix}$$

$$S_{zu} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \frac{1}{\sqrt{2}} \cdot S_{xu} + \frac{1}{\sqrt{2}} \cdot S_{xd} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad S_{zd} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \frac{1}{\sqrt{2}} \cdot S_{xu} - \frac{1}{\sqrt{2}} \cdot S_{xd} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The concept of the linear superposition is central in quantum theory and has no classical analog. For example, if by measurement an electron is found to have spin-up in the z-direction, this means that the electron does not have a definite spin in either the x- or the y-direction because $|S_{zu}\rangle$ is a linear superposition of the x- and y-direction spin states.

$$S_{zu} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \frac{1}{\sqrt{2}} \cdot S_{xu} + \frac{1}{\sqrt{2}} \cdot S_{xd} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \frac{1}{\sqrt{2}} \cdot S_{yu} + \frac{1}{\sqrt{2}} \cdot S_{yd} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

In spite of its appearance, **a linear superposition is not a mixture**. In other words $|S_{zu}\rangle$ is not 50% $|S_{xu}\rangle$ and 50% $|S_{xd}\rangle$, or 50% $|S_{yu}\rangle$ and 50% $|S_{yd}\rangle$.

Another central dogma of quantum theory is that the wavefunction or state vector contains all the physical information available for the system. Quantum mechanics therefore consists, in large part, of extracting physical information from the wavefunction or state vector. Quantum mechanics consists of a small set of rules for carrying this procedure out mathematically.

For every observable of the system there is an operator. Since electrons can spin in the x-, y-, or z-directions, there are spin operators in those directions, or for that matter in any other arbitrary direction you might think of. In quantum mechanics **states are vectors and operators are matrices.** The spin operators in units of h/4π are shown below. Note that squaring these operators gives the identity operator.

$$S_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S_x{}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad S_y{}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad S_z{}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

For example, the square of the total spin operator in units of h/4π is

$$S_{xyz} := S_x{}^2 + S_y{}^2 + S_z{}^2 \qquad S_{xyz} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

A measurement operator extracts information about the system by operating on the wavefunction or state vector. One possible outcome is that the operation returns the state vector multiplied by a numerical constant. For example,

$$S_x \cdot S_{xu} = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix} \qquad S_x \cdot S_{xd} = \begin{pmatrix} -0.707 \\ 0.707 \end{pmatrix} \qquad S_y \cdot S_{yu} = \begin{pmatrix} 0.707 \\ 0.707i \end{pmatrix} \qquad S_y \cdot S_{yd} = \begin{pmatrix} -0.707 \\ 0.707i \end{pmatrix}$$

$$S_z \cdot S_{zu} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad\qquad S_z \cdot S_{zd} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad \text{or, for example:} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

In Dirac notation we would summarize these calculations as follows: $\mathbf{S_x}|S_{xu}> = +1|S_{xu}>$, $\mathbf{S_x}|S_{xd}> = -1|S_{xd}>$, $\mathbf{S_y}|S_{yu}> = +1|S_{yu}>$, $\mathbf{S_y}|S_{yd}> = -1|S_{yd}>$, $\mathbf{S_z}|S_{zu}> = +1|S_{zu}>$, $\mathbf{S_z}|S_{zd}> = -1|S_{zd}>$. In each of these cases, the state vector is an eigenfunction of the measurement operator with eigenvalue of either +1 or -1 (in units of h/4π). We say, for example, that $|S_{xu}>$ is an eigenfunction of $\mathbf{S_x}$ with eigenvalue +1. The electron has a well-defined value for spin in the x-direction (spin-up) and subsequent measurements of the x-direction spin will yield the value of +1 as long as no intervening measurements in another spin direction are made.

The other possible outcome of the measurement operation is that it yields another state vector.

$$S_x \cdot S_{yu} = \begin{pmatrix} 0.707i \\ 0.707 \end{pmatrix} \qquad S_x \cdot S_{yd} = \begin{pmatrix} -0.707i \\ 0.707 \end{pmatrix} \qquad S_x \cdot S_{zu} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad S_x \cdot S_{zd} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$S_y \cdot S_{xu} = \begin{pmatrix} -0.707i \\ 0.707i \end{pmatrix} \qquad S_y \cdot S_{xd} = \begin{pmatrix} 0.707i \\ 0.707i \end{pmatrix} \qquad S_y \cdot S_{zu} = \begin{pmatrix} 0 \\ i \end{pmatrix} \qquad S_y \cdot S_{zd} = \begin{pmatrix} -i \\ 0 \end{pmatrix}$$

$$S_z \cdot S_{xu} = \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix} \qquad S_z \cdot S_{xd} = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix} \qquad S_z \cdot S_{yu} = \begin{pmatrix} 0.707 \\ -0.707i \end{pmatrix} \qquad S_z \cdot S_{yd} = \begin{pmatrix} 0.707 \\ 0.707i \end{pmatrix}$$

In Dirac notation these operations appear as: $\mathbf{S_x}|S_{yu}> = i|S_{yd}>$, $\mathbf{S_x}|S_{yd}> = -i|S_{yu}>$, $\mathbf{S_x}|S_{zu}> = |S_{zd}>$, $\mathbf{S_x}|S_{zd}> = |S_{zu}>$, etc. In each case the resulting vector is different than the vector operated on. We say, for example, $|S_{yu}>$ is not an eigenfunction of $\mathbf{S_x}$, and therefore an electron in this state does not have a definite value for spin in the x-direction. X-direction spin measurements on a system known to be in state $|S_{yu}>$ will yield completely random results.

To put it another way, quantum mechanical principles state that a system can be in a well-defined state, $|S_{yu}>$, and yet the outcome of all experiments are not uniquely determined. While a measurement of spin in the y-direction will yield a predictable result, +1, measurement of spin in the x- or z-direction is completely unpredictable and all we can calculate is the average value, or expectation value for a large number of measurements. This is completely different than classical physics where if you know the state of the system, you know the values of all physical observables.

As another example, consider the ground state of the hydrogen atom for which the electron's wave function is $\Psi = \pi^{-1/2}\exp(-r)$. When the electron is in this state it has a precise energy, but not a well-defined position or momentum. This, of course, makes the concept of an electron trajectory impossible and it is, therefore, meaningless to think of the electron as moving in any traditional sense.

The quantum mechanical algorithm for calculating the expectation value is to execute the following matrix multiplication: <State Vector|**Operator**|State Vector>. This formalism is quite general and can be used whether the state vector is an eigenfunction of the operator or not. This is demonstrated below for the spin states that we have been studying.

$$\left(\overline{S_{xu}}\right)^T \cdot S_x \cdot S_{xu} = 1 \qquad \left(\overline{S_{xd}}\right)^T \cdot S_x \cdot S_{xd} = -1 \qquad \left(\overline{S_{zu}}\right)^T \cdot S_x \cdot S_{zu} = 0$$

$$\left(\overline{S_{zd}}\right)^T \cdot S_x \cdot S_{zd} = 0 \qquad \left(\overline{S_{yu}}\right)^T \cdot S_x \cdot S_{yu} = 0 \qquad \left(\overline{S_{yd}}\right)^T \cdot S_x \cdot S_{yd} = 0$$

$$\left(\overline{S_{xu}}\right)^T \cdot S_y \cdot S_{xu} = 0 \qquad \left(\overline{S_{xd}}\right)^T \cdot S_y \cdot S_{xd} = 0 \qquad \left(\overline{S_{zu}}\right)^T \cdot S_y \cdot S_{zu} = 0$$

$$\left(\overline{S_{zd}}\right)^T \cdot S_y \cdot S_{zd} = 0 \qquad \left(\overline{S_{yu}}\right)^T \cdot S_y \cdot S_{yu} = 1 \qquad \left(\overline{S_{yd}}\right)^T \cdot S_y \cdot S_{yd} = -1$$

$$\left(\overline{S_{xu}}\right)^T \cdot S_z \cdot S_{xu} = 0 \qquad \left(\overline{S_{xd}}\right)^T \cdot S_z \cdot S_{xd} = 0 \qquad \left(\overline{S_{zu}}\right)^T \cdot S_z \cdot S_{zu} = 1$$

$$\left(\overline{S_{zd}}\right)^T \cdot S_z \cdot S_{zd} = -1 \qquad \left(\overline{S_{yu}}\right)^T \cdot S_z \cdot S_{yu} = 0 \qquad \left(\overline{S_{yd}}\right)^T \cdot S_z \cdot S_{yd} = 0$$

Let's look at the first six enteries because they are representative of the remaining results. If the electron is in the state $|S_{xu}>$ measurement of $\mathbf{S_x}$ will always yield the value of +1 (in units of h/4π). If the electron is in the state $|S_{xd}>$ measurement of Sx will always yield the value of -1 (in units of h/4π). If instead $\mathbf{S_y}$ or $\mathbf{S_z}$ are measured, the measurement results will be a statistically random collection of +1 and -1, and the average value will, of course, be zero. Only when the system is in an eigenstate of the measurement operator is the outcome of the experiment certain.

This brings us to the concept of probaility and how it is calculated in quantum mechanics. The projection of one state on to another, $<S_{zu}|S_{xd}>=.707$, is a **probability amplitude**. Its absolute square, $<S_{xd}|S_{zu}><S_{zu}|S_{xd}>=|<S_{zu}|S_{xd}>|^2=0.5$ (remember $<S_{xd}|S_{zu}>=<S_{zu}|S_{xd}>^*$), is the **probability** that an electron in state $|S_{xd}>$ will be found by measurement in the state $|S_{zu}>$. Representative calculations are shown below.

$$\left[\left|\left(\overline{S_{zu}}\right)^T \cdot S_{xu}\right|\right]^2 = 0.5 \qquad \left[\left|\left(\overline{S_{zd}}\right)^T \cdot S_{xu}\right|\right]^2 = 0.5 \qquad \left[\left|\left(\overline{S_{xu}}\right)^T \cdot S_{yu}\right|\right]^2 = 0.5 \qquad \left[\left|\left(\overline{S_{xu}}\right)^T \cdot S_{zd}\right|\right]^2 = 0.5$$

Let's review these concepts by taking a specific example. The electron is in the state $|S_{xu}\rangle$ and we wish to measure $\mathbf{S_z}$. According to quantum mechanical procedures the average value for a statistically meaningful number of measurements is zero - $\langle S_{xu}|\mathbf{S_z}|S_{xu}\rangle = 0$. The eigenstates (eigenfunctions) for $\mathbf{S_z}$ are $|S_{zu}\rangle$ and $|S_{zd}\rangle$ with eigenvalues +1 and -1, respectively. As the first two entries above show, the probability that an electron in state $|S_{xu}\rangle$ will be found in $|S_{zu}\rangle$ with eigenvalue +1 is 0.5, and the probaility that it will be found in state $|S_{zd}\rangle$ with eigenvalue -1 is 0.5. Thus, the **average value** is **expected** to be zero, and the two ways of determining the average or expectation value of a measurement are consistent and equivalent.

There is yet another way to look at this issue. In quantum mechanics for most pairs of observables the order of measurement is important. Quantum mechanical operators don't generally commute. For example, as shown below, $\mathbf{S_x S_y}|S_{zu}\rangle$ does not equal $\mathbf{S_y S_x}|S_{zu}\rangle$. This means that if the electron is in the state $|S_{zu}\rangle$ the combined operators $\mathbf{S_x S_y}$ and $\mathbf{S_y S_x}$ yield different measurement results.

$$S_x \cdot S_y \cdot S_{zu} = \begin{pmatrix} i \\ 0 \end{pmatrix} \qquad S_y \cdot S_x \cdot S_{zu} = \begin{pmatrix} -i \\ 0 \end{pmatrix} \qquad \left(S_x \cdot S_y - S_y \cdot S_x \right) \cdot S_{zu} = \begin{pmatrix} 2i \\ 0 \end{pmatrix}$$

Operators that do not commute have incompatible eigenstates. If a state vector is an eigenstate of one of the operators, it is not an eigenstate of the other. The fact that $\mathbf{Sx}$ and $\mathbf{Sy}$ do not commute means that an electron cannot simultaneously have well-defined values for S_x and S_y. It is not surprising that there is a deep connection between these properties of operators and the Uncertainty Principle. The commutators for the spin operators are shown below.

$$S_x \cdot S_y - S_y \cdot S_x = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} \qquad 2 \cdot i \cdot S_z = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} \qquad S_z \cdot S_x - S_x \cdot S_z = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

$$S_y \cdot S_z - S_z \cdot S_y = \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} \qquad 2 \cdot i \cdot S_x = \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} \qquad 2 \cdot i \cdot S_y = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

The Uncertainty Principle can also be illustrated by calculating ΔSx and ΔSy for an electron known to be in the Szu state. Since we are working in units of $h/4\pi$, the uncertainty relation is: $\Delta S_x \cdot \Delta S_y \geq 1$.

$$\sqrt{ S_{zu}{}^T \cdot S_x \cdot S_x \cdot S_{zu} - \left(S_{zu}{}^T \cdot S_x \cdot S_{zu} \right)^2 } \cdot \sqrt{ S_{zu}{}^T \cdot S_y \cdot S_y \cdot S_{zu} - \left(S_{zu}{}^T \cdot S_y \cdot S_{zu} \right)^2 } = 1$$

We have been dealing with matrix operators and their associated eigenvectors and eigenvalues. The eigenvectors and eigenvalues can be obtained from the matrix operators with Mathcad's **eigenvecs** and **eigenvals** commands as is shown below.

$$eigenvals\left(S_x \right) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad eigenvec\left(S_x, 1 \right) = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix} \qquad eigenvec\left(S_x, -1 \right) = \begin{pmatrix} -0.707 \\ 0.707 \end{pmatrix}$$

$$eigenvals\left(S_y \right) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad eigenvecs\left(S_y \right) = \begin{pmatrix} -0.707i & 0.707 \\ 0.707 & -0.707i \end{pmatrix}$$

$$eigenvals\left(S_z \right) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad eigenvecs\left(S_z \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Matrix Mechanics of Spin States: Two-photon Interference

Reference: Frank Rioux
Reference: Greenberger, D.M.; Horne, M.A.; Zeilinger, A. *Physics Today*, **1993**, *44(8)*, 22.

In this experiment a down converter, DC, transforms an incident photon into two lower energy photons. One photon takes the upper path and the other the lower path or vice versa. The results of this experiment are that both photons are detected at either U or D. One photon is never detected at U while the other is detected at D. A quantum mechanical analysis of this phenomena is provided below.

```
      UP

   M ------    M = mirror
     ^
    /\
   / \   U-detector
  / \ /
 /  \/
Source -->DC   BS -x--- DC = down converter  BS = 50/50 beam splitter
  \ / \
   \ /  D-detector
    \/
    v
  M ------    M = mirror


     DOWN
```

Orthonormal basis states:

Photon moving in up-direction:
$$u := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (u)^T \cdot u = 1$$

Photon moving in down-direction:
$$d := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (d)^T \cdot d = 1 \quad (u)^T \cdot d = 0$$

Operators:

Operator for interaction with the mirror:
$$M := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Operator for interaction with a 50/50 beam splitter:
$$BS := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

A 90° phase shift between transmission and reflection at the beam splitter is required to satisfy energy conservation. By convention the phase shift is assigned to reflection.

The down-converter creates the following entangled state:

$$|\Psi_b\rangle = [|u\rangle_1 |d\rangle_2 + |d\rangle_1 |u\rangle_2]/2^{1/2}$$

This is a symmetric state because photons are bosons.

After creation in the down-converter, both photons interact with a mirror and a beam splitter before reaching a detector, either U or D. To be detected at the U-detector the photon must be moving in the up-direction (photon state = $|u>$). To be detected at the D-detector the photon must be moving in the down-direction (photon state = $|d>$). The probabilities for the four possible experimental outcomes are calculated below.

Both photons arrive at the U-detector: $|_1<u|_2<u|\textbf{BS M}|\Psi_b>|^2$

$$\left[\left|\left|\frac{\left[(u)^T \cdot BS \cdot M \cdot u\right] \cdot \left[(u)^T \cdot BS \cdot M \cdot d\right] + \left[(u)^T \cdot BS \cdot M \cdot d\right] \cdot \left[(u)^T \cdot BS \cdot M \cdot u\right]}{\sqrt{2}}\right|\right|\right]^2 = 0.5$$

Both photons arrive at the D-detector: $|_1<d|_2<d|\textbf{BS M}|\Psi_b>|^2$

$$\left[\left|\left|\frac{\left[(d)^T \cdot BS \cdot M \cdot u\right] \cdot \left[(d)^T \cdot BS \cdot M \cdot d\right] + \left[(d)^T \cdot BS \cdot M \cdot d\right] \cdot \left[(d)^T \cdot BS \cdot M \cdot u\right]}{\sqrt{2}}\right|\right|\right]^2 = 0.5$$

Photon 1 arrives at the U-detector and photon 2 arrives at the D-detector: $|_1<u|_2<d|\textbf{BS M}|\Psi_b>|^2$

$$\left[\left|\left|\frac{\left[(u)^T \cdot BS \cdot M \cdot u\right] \cdot \left[(d)^T \cdot BS \cdot M \cdot d\right] + \left[(u)^T \cdot BS \cdot M \cdot d\right] \cdot \left[(d)^T \cdot BS \cdot M \cdot u\right]}{\sqrt{2}}\right|\right|\right]^2 = 0$$

Photon 1 arrives at the D-detector and photon 2 arrives at the U-detector: $|_1<d|_2<u|\textbf{BS M}|\Psi_b>|^2$

$$\left[\left|\left|\frac{\left[(d)^T \cdot BS \cdot M \cdot u\right] \cdot \left[(u)^T \cdot BS \cdot M \cdot d\right] + \left[(d)^T \cdot BS \cdot M \cdot d\right] \cdot \left[(u)^T \cdot BS \cdot M \cdot u\right]}{\sqrt{2}}\right|\right|\right]^2 = 0$$

If the experiment could be performed with fermions, they would be created in the following anti-symmetric entangled state:

$$|\Psi_f> = [|u>_1|d>_2 - |d>_1|u>_2]/2^{1/2}$$

As the analysis below shows, the results for fermions would be exactly opposite to those for bosons. Two fermions would never arrive at the same detector.

Both fermions arrive at the U-detector: $|_1{<}u|_2{<}u|\mathbf{BS\ M}|\Psi_f{>}|^2$

$$\left[\left\|\frac{\left[(u)^T\cdot BS\cdot M\cdot u\right]\cdot\left[(u)^T\cdot BS\cdot M\cdot d\right]-\left[(u)^T\cdot BS\cdot M\cdot d\right]\cdot\left[(u)^T\cdot BS\cdot M\cdot u\right]}{\sqrt{2}}\right\|\right]^2 = 0$$

Both fermions arrive at the D-detector: $|_1{<}d|_2{<}d|\mathbf{BS\ M}|\Psi_f{>}|^2$

$$\left[\left\|\frac{\left[(d)^T\cdot BS\cdot M\cdot u\right]\cdot\left[(d)^T\cdot BS\cdot M\cdot d\right]-\left[(d)^T\cdot BS\cdot M\cdot d\right]\cdot\left[(d)^T\cdot BS\cdot M\cdot u\right]}{\sqrt{2}}\right\|\right]^2 = 0$$

Fermion 1 arrives at the U-detector and fermion 2 arrives at the D-detector: $|_1{<}u|_2{<}d|\mathbf{BS\ M}|\Psi_f{>}|^2$

$$\left[\left\|\frac{\left[(u)^T\cdot BS\cdot M\cdot u\right]\cdot\left[(d)^T\cdot BS\cdot M\cdot d\right]-\left[(u)^T\cdot BS\cdot M\cdot d\right]\cdot\left[(d)^T\cdot BS\cdot M\cdot u\right]}{\sqrt{2}}\right\|\right]^2 = 0.5$$

Fermion 1 arrives at the D-detector and fermion 2 arrives at the U-detector: $|_1{<}d|_2{<}u|\mathbf{BS\ M}|\Psi_f{>}|^2$

$$\left[\left\|\frac{\left[(d)^T\cdot BS\cdot M\cdot u\right]\cdot\left[(u)^T\cdot BS\cdot M\cdot d\right]-\left[(d)^T\cdot BS\cdot M\cdot d\right]\cdot\left[(u)^T\cdot BS\cdot M\cdot u\right]}{\sqrt{2}}\right\|\right]^2 = 0.5$$

The results of this tutorial enable us to formulate a sociology for bosons and fermions: bosons are gregarious and enjoy companionship; fermions are anti-social and prefer solitude.

But why do bosons always end up at the same detector and fermions (hypothetically) always end up at different detectors? Why in both cases do half of the possible outcomes not occur? Are the bosons and fermions interfering with each other directly? Is there a subtle attractive interaction between bosons and an equally subtle, non-electrostatic, repulsive interaction between fermions?

Not according to Roy Glauber who said, "**The things that interfere in quantum mechanics are not particles**. They are probability amplitudes for certain events. It is the fact that **probability amplitudes add up like complex numbers that accounts for all quantum mechanical interferences**."
 [American Journal of Physics **63**, 12 (1995)]

The analysis used in this tutorial clearly illustrates Glauber's assertion.

Reference: Greenberger, D.M.; Horne, M.A.; Zeilinger, A. *Physics Today*, **1993**, *44(8)*, 22.

Tensor Products of Vector Spaces

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The *tensor product* $V \otimes W$ of two vector spaces V and W with bases $A = \{|\alpha_1\rangle, |\alpha_2\rangle, \ldots, |\alpha_n\rangle\}$ and $B = \{|\beta_1\rangle, |\beta_2\rangle, \ldots, |\beta_m\rangle\}$ respectively is an nm-dimensional vector space with a basis consisting of the nm elements of the form $|\alpha_i\rangle \otimes |\beta_j\rangle$ where $\otimes$ is the tensor product, an abstract binary operator that satisfies the following relations:

$$(|v_1\rangle + |v_2\rangle) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle$$

$$|v\rangle \otimes (|w_1\rangle + |w_2\rangle) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle$$

$$(a|v\rangle) \otimes |w\rangle = |v\rangle \otimes (a|w\rangle) = a(|v\rangle \otimes |w\rangle).$$

Taking $k = \min(n, m)$, all elements of $V \otimes W$ have form

$$|v_1\rangle \otimes |w_1\rangle + |v_2\rangle \otimes |w_2\rangle + \cdots + |v_k\rangle \otimes |w_k\rangle$$

for some $v_i \in V$ and $w_i \in W$. Due to the relations defining the tensor product, such a representation is not unique. Furthermore, while all elements of $V \otimes W$ can be written

$$\alpha_1(|\alpha_1\rangle \otimes |\beta_1\rangle) + \alpha_2(|\alpha_2\rangle \otimes |\beta_1\rangle) + \cdots + \alpha_{nm}(|\alpha_n\rangle \otimes |\beta_m\rangle),$$

most elements of $V \otimes W$ cannot be written as $|v\rangle \otimes |w\rangle$, where $v \in V$ and $w \in W$. It is common to write $|v\rangle|w\rangle$ for $|v\rangle \otimes |w\rangle$.

VII. A Proof of Bell's Theorem

This Analysis is Based Jim Baggott's analysis of Bell's theorem as presented in Chapter 4 of *The Meaning of Quantum Theory* and Frank Rioux's Methodology using matrix and tensor algebra.

A two-stage atomic cascade emits entangled photons (A and B) in opposite directions with the same circular polarization according to observers in their path.

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|L\rangle_A|L\rangle_B + |R\rangle_A|R\rangle_B\right]$$

The experiment involves the measurement of photon polarization states in the vertical/horizontal measurement basis, and allows for the rotation of the right-hand detector through an angle of θ, in order to explore the consequences of quantum mechanical entanglement. PA stands for polarization analyzer and could simply be a calcite crystal.
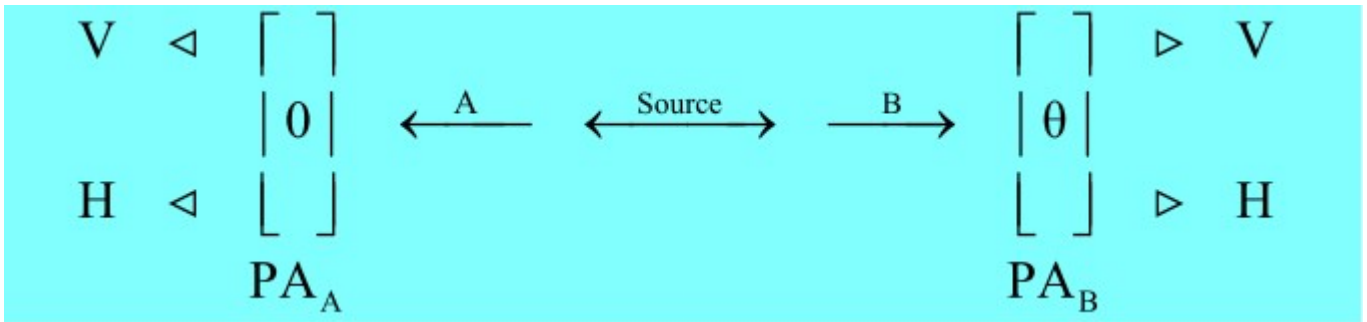
In vector notation the left- and right-circular polarization states are expressed as follows:

Left circular polarization: $\quad L := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\i\end{pmatrix}$
Right circular polarization: $\quad R := \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\-i\end{pmatrix}$

In tensor notation the initial state is the following entangled superposition is,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|L\rangle_A|L\rangle_B + |R\rangle_A|R\rangle_B\right] = \frac{1}{2\sqrt{2}}\left[\begin{pmatrix}1\\i\end{pmatrix}_A \otimes \begin{pmatrix}1\\i\end{pmatrix}_B + \begin{pmatrix}1\\-i\end{pmatrix}_A \otimes \begin{pmatrix}1\\-i\end{pmatrix}_B\right] = \frac{1}{2\sqrt{2}}\left[\begin{pmatrix}1\\i\\i\\-1\end{pmatrix} + \begin{pmatrix}1\\-i\\-i\\-1\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$$

However, as mentioned above, the photon polarization measurements will actually be made in the vertical/horizontal basis. These polarization measurement states for photons A and B in vector representation are given below. Θ is the angle through which the PA_2 has been rotated.

Vertical Polarization: $\quad V_A := \begin{pmatrix}1\\0\end{pmatrix} \quad V_B := \begin{pmatrix}\cos(\theta)\\-\sin(\theta)\end{pmatrix}$
Horizontal Polarization: $\quad H_A := \begin{pmatrix}0\\1\end{pmatrix} \quad H_B := \begin{pmatrix}\sin(\theta)\\\cos(\theta)\end{pmatrix}$

It is easy to show that $|\Psi\rangle$ in the vertical/horizontal basis is,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|V\rangle_A|V\rangle_B - |H\rangle_A|H\rangle_B\right] = \frac{1}{2\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}_A \otimes \begin{pmatrix}1\\0\end{pmatrix}_B - \begin{pmatrix}0\\1\end{pmatrix}_A \otimes \begin{pmatrix}0\\1\end{pmatrix}_B\right] = \frac{1}{2\sqrt{2}}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix} - \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$$

There are four possible measurement outcomes: both photons are vertically polarized, both are horizontally polarized, one is vertical and the other horizontal, and vice versa. The vector representations of the measurement states are obtained by tensor multiplication of the individual photon states.

$$|V_A V_B\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta \\ -\sin\theta \\ 0 \\ 0 \end{pmatrix} \qquad |V_A H_B\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \sin\theta \\ \cos\theta \\ 0 \\ 0 \end{pmatrix}$$

$$|H_A V_B\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cos\theta \\ -\sin\theta \end{pmatrix} \qquad |H_A H_B\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sin\theta \\ \cos\theta \end{pmatrix}$$

The initial state and the measurement eigenstates are written in Mathcad syntax.

$$\Psi := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \qquad VaVb(\theta) := \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \\ 0 \\ 0 \end{pmatrix} \qquad VaHb(\theta) := \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \\ 0 \\ 0 \end{pmatrix}$$

$$HaVb(\theta) := \begin{pmatrix} 0 \\ 0 \\ \cos(\theta) \\ -\sin(\theta) \end{pmatrix} \qquad HaHb(\theta) := \begin{pmatrix} 0 \\ 0 \\ \sin(\theta) \\ \cos(\theta) \end{pmatrix}$$

<u>The projections of the initial state onto the four measurement states are,</u>

Note: θ.dot --> $\quad \theta := \pi$

Probability Amplitude:
$$\begin{pmatrix} VaVb(\theta)^T \cdot \Psi \\ VaHb(\theta)^T \cdot \Psi \\ HaVb(\theta)^T \cdot \Psi \\ HaHb(\theta)^T \cdot \Psi \end{pmatrix} \rightarrow \begin{pmatrix} -\dfrac{\sqrt{2}}{2} \\ 0 \\ 0 \\ \dfrac{\sqrt{2}}{2} \end{pmatrix}$$

Probability:
$$\begin{bmatrix} \left(VaVb(\theta)^T \cdot \Psi\right)^2 \\ \left(VaHb(\theta)^T \cdot \Psi\right)^2 \\ \left(HaVb(\theta)^T \cdot \Psi\right)^2 \\ \left(HaHb(\theta)^T \cdot \Psi\right)^2 \end{bmatrix} \rightarrow \begin{pmatrix} \dfrac{1}{2} \\ 0 \\ 0 \\ \dfrac{1}{2} \end{pmatrix}$$

Assigning an eigenvalue of +1 to a vertical polarization measurement and -1 to a horizontal polarization measurement allows the calculation of the expectation value for the joint polarization measurements, a function which quantifies the correlation between the joint measurements. The eigenvalues for the four joint measurement outcomes are: VaVb = 1; VaHb = -1; HaVb = -1; HaHb = 1. Weighting these by the probability of their occurence gives the **Expectation Value, E(θ)** or **Correlation Function.**

$$E(\theta) := \left(VaVb(\theta)^T \cdot \Psi\right)^2 - \left(VaHb(\theta)^T \cdot \Psi\right)^2 - \left(HaVb(\theta)^T \cdot \Psi\right)^2 + \left(HaHb(\theta)^T \cdot \Psi\right)^2 \rightarrow \cos(\theta)^2 - \sin(\theta)^2$$

$$\rightarrow \cos(2 \cdot \theta)$$

As shown above the evaluation of E(θ) yields cos(2θ). For θ = 0⁰ there is perfect correlation; for θ = 90⁰ perfect anti-correlation; for θ = 45⁰ no correlation.

$$E(0 \cdot deg) = 1 \qquad\qquad E(90 \cdot deg) = -1 \qquad\qquad E(45 \cdot deg) = 0$$

Baggott presented a correlation function for this experiment based on a local hidden variable model of reality (pp. 110-113, 127-131). It (linear blue line) and the quantum mechanical correlation function, E(θ), are compared on the graph below. Quantum theory and local realism disagree at all angles except 0, 45, and 90 degrees.

Comparison Quantum Theory Expectation vs. Local Hidden Variable

Bell's Theorem Correlation Function
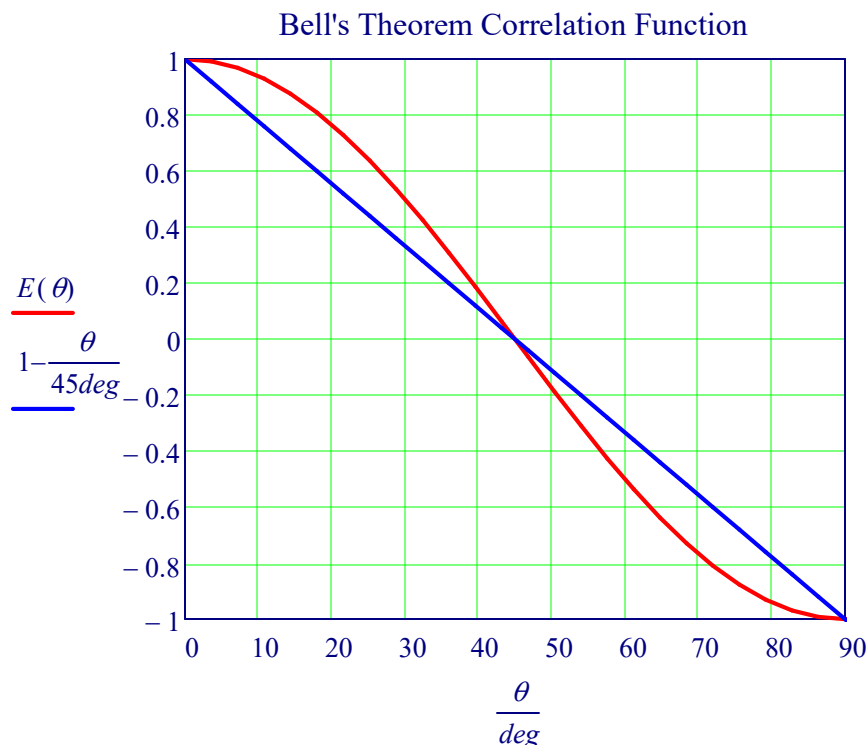
VIII. "Qubit Quantum Mechanics with Correlated-Photon Experiments"

Matrix Based Formalism to Model Some Classic QM Experiments, Paper by Galvez, AJP 78, 510-519 (2010)

**This document uses the Mathcad programming environment to model and reproduce
most of the results for the experiments presented in Professor Galvez's paper.**

Polarization Space and Direction of Propagation: State Vectors

Photon moving horizontally:
$$x := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Photon moving vertically:
$$y := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Null vector:
$$n := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Horizontal polarization:
$$h := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Vertical polarization:
$$v := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Diagonal polarization:
$$d := \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Single mode operators:

Projection operators for motion in the x- and y-directions:
$$X := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad Y := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Operator for polarizing film oriented at angle of θ to the horizontal.
$$\Theta_{op}(\theta) := \begin{pmatrix} cos(\theta) \\ sin(\theta) \end{pmatrix} \cdot \begin{pmatrix} cos(\theta) & sin(\theta) \end{pmatrix} \rightarrow \begin{pmatrix} cos(\theta)^2 & cos(\theta) \cdot sin(\theta) \\ cos(\theta) \cdot sin(\theta) & sin(\theta)^2 \end{pmatrix}$$

Beam splitter:
$$BS := \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Mirror:
$$M := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Phase shift:
$$A(\delta) := \begin{pmatrix} e^{i \cdot \delta} & 0 \\ 0 & 1 \end{pmatrix}$$

Half and quarter wave plate:
$$W_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad W_4 := \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$$

Identity:
$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Rotated half wave plate:
$$W_2(\theta) := \begin{pmatrix} cos(2 \cdot \theta) & sin(2 \cdot \theta) \\ sin(2 \cdot \theta) & -cos(2 \cdot \theta) \end{pmatrix}$$

<u>Equation 17</u>
$$W(\theta) := \begin{pmatrix} cos(2 \cdot \theta) & sin(2 \cdot \theta) & 0 & 0 \\ sin(2 \cdot \theta) & -cos(2 \cdot \theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Mach-Zehnder interferometer:
$$MZ(\delta) := BS \cdot A(\delta) \cdot M \cdot BS$$

Two mode states and operators:

Single-photon direction of propagation and polarization states Eq 15:

$$xh := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad xv := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad yh := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad yv := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Two-photon direction of propagation states.

$$xx := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad xy := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad yx := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad yy := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Two-photon polarization states.

$$hh := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad hv := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad vh := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad vv := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Polarizing Beam Splitter, PBS, which transmits horizontally polarized photons and reflects vertically polarized photons. Eq. 24

$$PBS := xh \cdot xh^T + yv \cdot xv^T + yh \cdot yh^T + xv \cdot yv^T \qquad PBS = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Polarization M-Z interferometer: $\qquad MZ_P(\delta) := PBS \cdot kronecker(A(\delta), I) \cdot kronecker(M, I) \cdot PBS$

Kronecker is Mathcad's command for tensor multiplication of square matrices.

Mach-Zehnder interferometer for direction of propagation and polarization, which places a rotatable half-wave plate in the upper path. Eq. 24

$$MZ_{dp}(\theta, \delta) := kronecker(BS, I) \cdot kronecker(A(\delta), I) \cdot W(\theta) \cdot kronecker(M, I) \cdot kronecker(BS, I)$$

Mach-Zehnder two-photon direction-of-propagation interferometer.

$$AA(\delta) := kronecker(A(\delta), A(\delta))$$

$$BSBS := kronecker(BS, BS) \qquad MM := kronecker(M, M)$$

$$MZ_{dd}(\delta) := BSBS \cdot AA(\delta) \cdot MM \cdot BSBS$$

The results of single photons going through the interferometer and being detected at the two outputs of the interferometer are shown in Fig. 2.

Confirm the results in Figure 2 for the Mach-Zehnder interferometer:

$$\delta := 0, .125 \cdot \pi .. 6\pi$$

$$\left(\left| x^T \cdot MZ(\delta) \cdot x \right| \right)^2$$

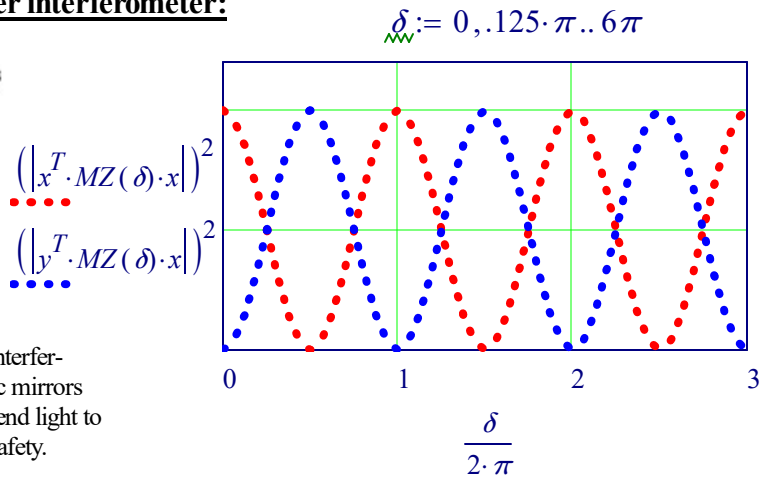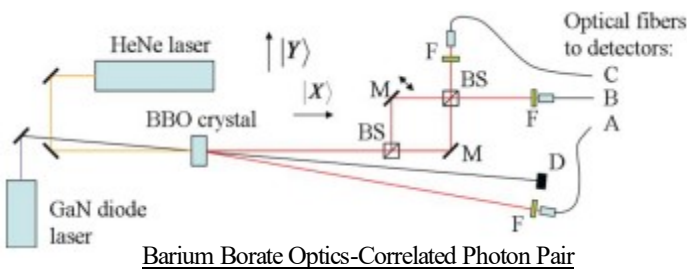$$\left(\left| y^T \cdot MZ(\delta) \cdot x \right| \right)^2$$

Fig. 1. Standard layout for doing experiments with correlated photons. Interferometer components are **nonpolarizing beam splitters (BS)** and metallic mirrors (m). Band-pass filters (f) precede couplers to multimode fibers, which send light to detectors A, B, and C. The beam dump (d) collects the pump beam for safety.
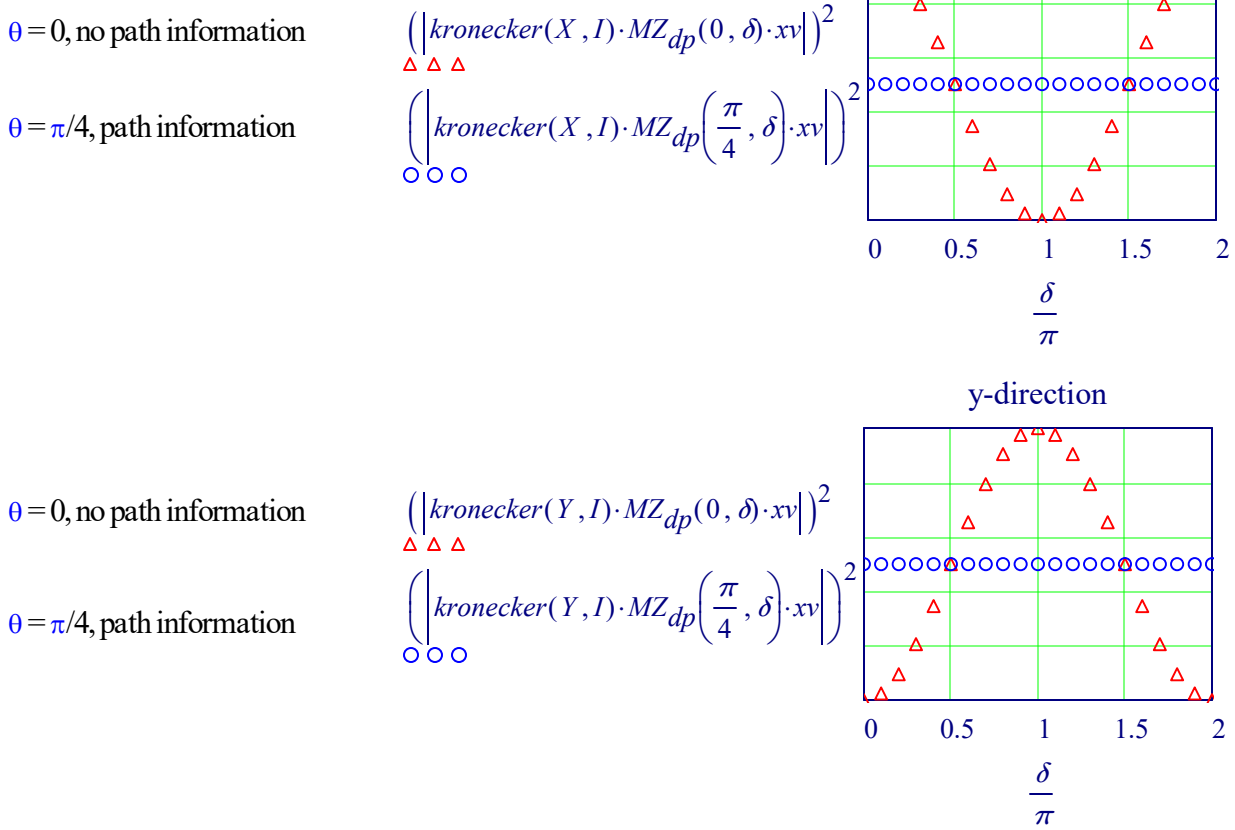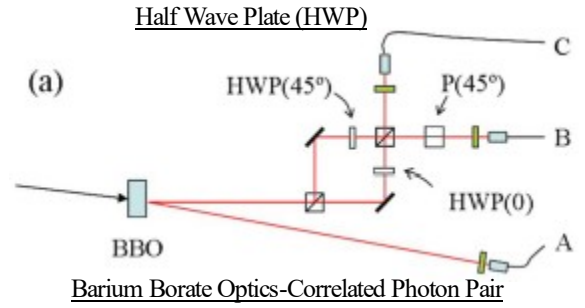
$$\frac{\delta}{2 \cdot \pi}$$

Demonstrate that a superposition is formed after first beam splitter

$$BS \cdot x = \begin{pmatrix} 0.707 \\ 0.707i \end{pmatrix} \qquad \frac{1}{\sqrt{2}} \cdot (x + i \cdot y) = \begin{pmatrix} 0.707 \\ 0.707i \end{pmatrix}$$

$$\delta := 0, .1 \cdot \pi .. 2 \cdot \pi$$

Confirmation that <u>path information destroys interference.</u>

(a)

Half Wave Plate (HWP)

Barium Borate Optics-Correlated Photon Pair

Fig. 3. Schematic of the (a) apparatus and (b) data for the quantum eraser. The data show cases when the light leave⬚ direction not carrying path information (tr⬚ along the Y direction carrying path inform⬚

$\theta = 0$, no path information

$$\left(\left| kronecker(X, I) \cdot MZ_{dp}(0, \delta) \cdot xv \right| \right)^2$$

$\theta = \pi/4$, path information

$$\left(\left| kronecker(X, I) \cdot MZ_{dp}\left(\frac{\pi}{4}, \delta\right) \cdot xv \right| \right)^2$$

x-direction

$\theta = 0$, no path information

$$\left(\left| kronecker(Y, I) \cdot MZ_{dp}(0, \delta) \cdot xv \right| \right)^2$$

$\theta = \pi/4$, path information

$$\left(\left| kronecker(Y, I) \cdot MZ_{dp}\left(\frac{\pi}{4}, \delta\right) \cdot xv \right| \right)^2$$

y-direction

Erasure of path information restores interference. Erasers for the x- and y-directions place diagonal polarizers in those directions after the interferometer.

$$E_x := \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad E_y := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$
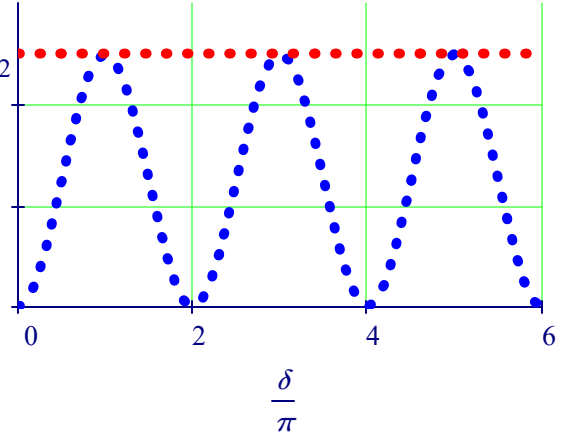
The x-direction has an eraser and the y-direction does not. $\qquad$ $\delta := 0, .1 \cdot \pi .. 6\pi$

x-direction: $\qquad$ $\left(\left\| kronecker(X, I) \cdot E_x \cdot MZ_{dp}\left(\dfrac{\pi}{4}, \delta\right) \cdot xv \right\| \right)^2$
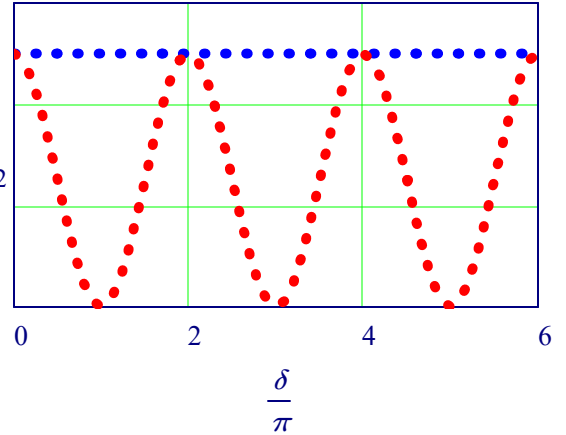
$\bullet \ \bullet \ \bullet \ \bullet$

y-direction: $\qquad$ $\left(\left\| kronecker(Y, I) \cdot MZ_{dp}\left(\dfrac{\pi}{4}, \delta\right) \cdot xv \right\| \right)^2$

$\color{red}{\bullet \ \bullet \ \bullet \ \bullet}$

The y-direction has an eraser and the x-direction does not.

x-direction: $\qquad$ $\left(\left\| kronecker(X, I) \cdot MZ_{dp}\left(\dfrac{\pi}{4}, \delta\right) \cdot xv \right\| \right)^2$

$\bullet \ \bullet \ \bullet \ \bullet$

y-direction: $\qquad$ $\left(\left\| kronecker(Y, I) \cdot E_y \cdot MZ_{dp}\left(\dfrac{\pi}{4}, \delta\right) \cdot xv \right\| \right)^2$

$\color{red}{\bullet \ \bullet \ \bullet \ \bullet}$

For the MZ polarization interferometer diagonally polarized light enters in the x-direction, $|xd>$.
Tensor vector multiplication is awkward in Mathcad as is shown below.

$$\Psi_{in} := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad\qquad submatrix(kronecker(augment(x, n), augment(d, n)), 1, 4, 1, 1) = \blacksquare$$

No light, however, exits in the x-direction. It exits in the y-direction showing no interference effects.

$\delta := 0, .2 \cdot \pi .. \pi$

$$\left(\left\| kronecker(X, I) \cdot MZ_P(\delta) \cdot \Psi_{in} \right\| \right)^2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad\qquad \left(\left\| kronecker(Y, I) \cdot MZ_P(\delta) \cdot \Psi_{in} \right\| \right)^2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Placement of a **D** polarizer in the y-direction output erases distinguishing information and interference appears.
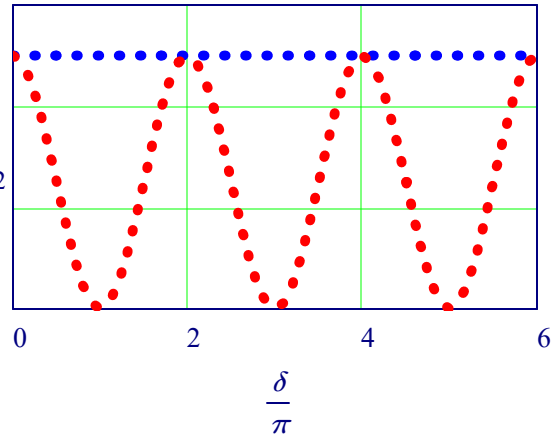
$$\delta := 0, .1 \cdot \pi .. 6\pi$$

x-direction:

$$\left(\left\|kronecker(X, I) \cdot MZ_{dp}\left(\frac{\pi}{4}, \delta\right) \cdot xv\right\|\right)^2$$

• • • •

y-direction:

$$\left(\left\|kronecker(Y, I) \cdot E_y \cdot MZ_{dp}\left(\frac{\pi}{4}, \delta\right) \cdot xv\right\|\right)^2$$
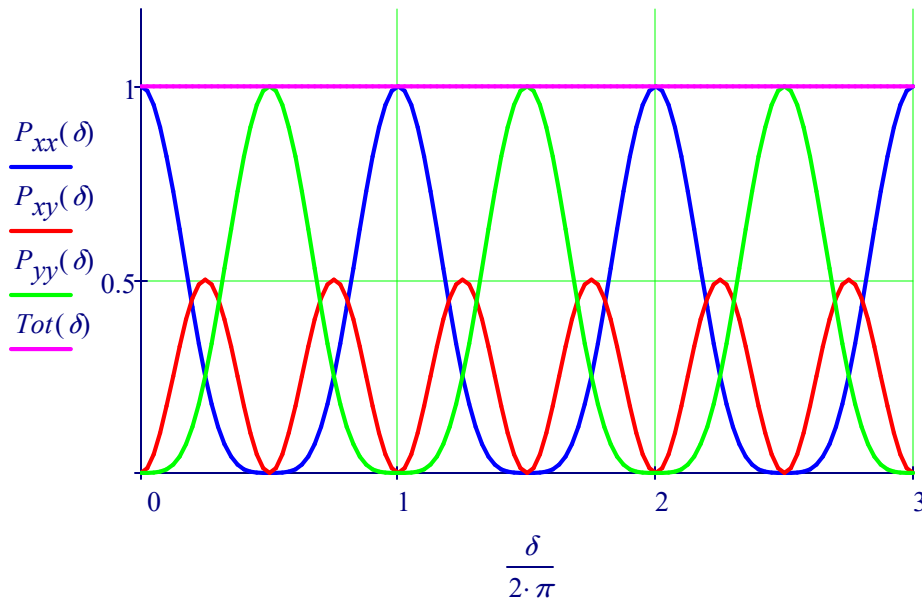
• • • •

Calculation of exit probabilities for two photons in direction-of-propagation modes:

$$P_{xx}(\delta) := \left(\left\|xx^T \cdot MZ_{dd}(\delta) \cdot xx\right\|\right)^2 \qquad P_{xy}(\delta) := \left[\left\|\frac{1}{\sqrt{2}} \cdot (xy + yx)^T \cdot MZ_{dd}(\delta) \cdot xx\right\|\right]^2$$

$$P_{yy}(\delta) := \left(\left\|yy^T \cdot MZ_{dd}(\delta) \cdot xx\right\|\right)^2 \qquad Tot(\delta) := P_{xx}(\delta) + P_{xy}(\delta) + P_{yy}(\delta)$$

Reproduction of Figure 5b with the addition of P_{yy}.

$$\delta := 0, .05 \cdot \pi .. 6 \cdot \pi$$

"The striking result is that the (P_{xy}) interference pattern has twice the frequency of the single-photon interference pattern. Nonclassical interference shows new quantum aspects: two photons acting as a single quantum object (a biphoton)."

Hong-Ou-Mandel interference
(right column, page 516):

$$BSBS \cdot \frac{1}{\sqrt{2}} \cdot (xy + yx) = \begin{pmatrix} 0.707i \\ 0 \\ 0 \\ 0.707i \end{pmatrix} \qquad \frac{i}{\sqrt{2}} \cdot (xx + yy) = \begin{pmatrix} 0.707i \\ 0 \\ 0 \\ 0.707i \end{pmatrix}$$

Section III.D deals with distinguishing between pure and mixed states experimentally. The pure state and it density matrix are given below.

$$\Psi_{pure} := \frac{1}{\sqrt{2}} \cdot (hh + vv) \qquad \Psi_{pure} = \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ 0.707 \end{pmatrix} \qquad \Psi_{pure} \cdot \Psi_{pure}^T = \begin{pmatrix} 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 \end{pmatrix}$$

The density matrix for the mixed state is calculated as follows.

$$\frac{1}{2} \cdot hh \cdot hh^T + \frac{1}{2} \cdot vv \cdot vv^T = \begin{pmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 \end{pmatrix}$$

The following calculations and their graphical representation are in complete agreement with section III.D

$$Pure(\alpha) := tr\left[\frac{1}{2} \cdot \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \cdot \frac{1}{2} \cdot \left[\begin{pmatrix} cos(\alpha) \\ cos(\alpha) \\ sin(\alpha) \\ sin(\alpha) \end{pmatrix} \cdot \begin{pmatrix} cos(\alpha) \\ cos(\alpha) \\ sin(\alpha) \\ sin(\alpha) \end{pmatrix}^T \right] \right] \; simplify \; \rightarrow \; \frac{sin(2 \cdot \alpha)}{4} + \frac{1}{4}$$

$$Mixed(\alpha) := tr\left[\frac{1}{2} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \frac{1}{2} \cdot \left[\begin{pmatrix} cos(\alpha) \\ cos(\alpha) \\ sin(\alpha) \\ sin(\alpha) \end{pmatrix} \cdot \begin{pmatrix} cos(\alpha) \\ cos(\alpha) \\ sin(\alpha) \\ sin(\alpha) \end{pmatrix}^T \right] \right] \; simplify \; \rightarrow \; \frac{1}{4}$$

Reproduce Figure 6 results. $\alpha := 0 \cdot deg, 5 \cdot deg .. 180 \cdot deg$

$Pure(\alpha)$
$\circ\circ\circ$
$Mixed(\alpha)$
$\triangle\triangle\triangle$

$$\frac{\alpha}{deg}$$
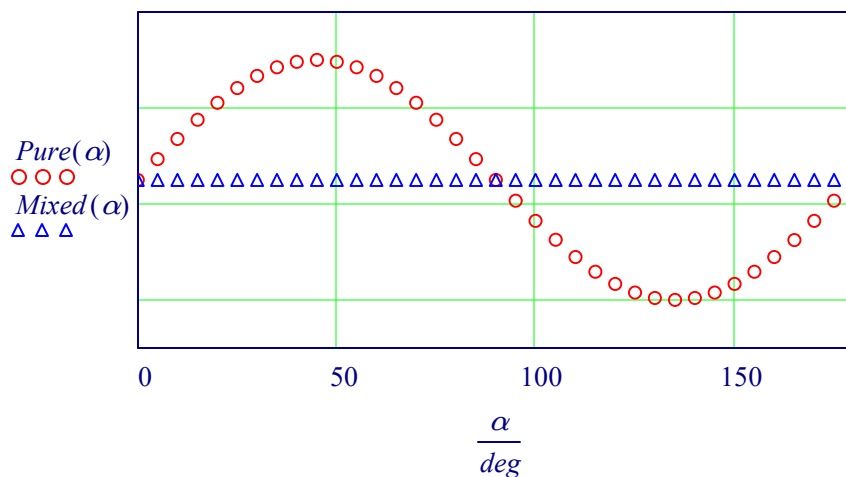
The following calculation are in agreement with the math in the final paragraph of section IV.D.

$$kronecker\left(W_2(0),I\right) \cdot \Psi_{pure} = \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ -0.707 \end{pmatrix} \quad \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ -0.707 \end{pmatrix} \cdot \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ -0.707 \end{pmatrix}^T = \begin{pmatrix} 0.5 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0 & 0.5 \end{pmatrix}$$

$$kronecker\left(W_2(0),I\right) \cdot \Psi_{pure} \cdot \Psi_{pure}^T \cdot kronecker\left(W_2(0),I\right)^T = \begin{pmatrix} 0.5 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0 & 0.5 \end{pmatrix}$$

$$Pure(\alpha) := tr\left[\frac{1}{2} \cdot \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \cdot \frac{1}{2} \cdot \left[\begin{pmatrix} cos(\alpha) \\ cos(\alpha) \\ sin(\alpha) \\ sin(\alpha) \end{pmatrix} \cdot \begin{pmatrix} cos(\alpha) \\ cos(\alpha) \\ sin(\alpha) \\ sin(\alpha) \end{pmatrix}^T\right]\right] \quad simplify \rightarrow \frac{1}{4} - \frac{sin(2\cdot\alpha)}{4}$$

The Galvez paper shows this as [1 - sin(α)]/4, which is a typographical error. The correct answer is [1 - sin(2*α)]/4
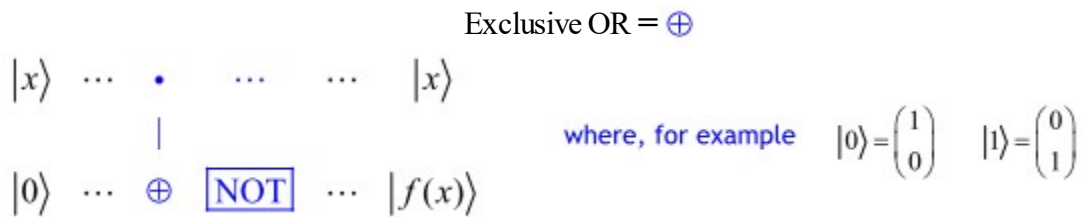
IX. Simple Example of Parallel Quantum Computation

This tutorial deals with quantum function evaluation and parallel computation. The example is taken from pages 94-95 of *Exploring the Quantum* by Haroche and Raimond. A certain function of x yields the following table of results.

Computing a function f(x) with a quantum machine

$$\begin{pmatrix} x & 0 & 1 \\ f(x) & 1 & 0 \end{pmatrix}$$

First we establish that the **circuit shown below** yields the results given in the table, and then demonstrate that it also **carries out a parallel calculation** in one step using both input values of x.

Exclusive OR = $\oplus$

where, for example $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The top wire carries the value of x and the bottom wire is initially set to |0>. After operation of the controlled-NOT and NOT gates, x remains on the top wire while the bottom wire carries the value of the function, f(x). In other words,

$$\hat{U}_f |x\rangle |0\rangle = |x\rangle |f(x)\rangle$$

The quantum gates in matrix form are: $I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $NOT := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $CNOT := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Uf (controlled-NOT, followed by a NOT operation on the lower wire) is a reversible operator. Doing it twice in succession on the initial two-qubit state is equivalent to the identity operation.
Note that the identity operator is required when a wire is not involved in an operation. In what follows the quantum circuit is constructed, displayed and its reversibility demonstrated. In other words, repeating the circuit is equivalent to the identity operation. Reversibility is a crucial property in quantum computer circuitry.

Kronecker is Mathcad's command for carrying out matrix tensor multiplication.

$$QuantumCircuit := kronecker(I, NOT) \cdot CNOT$$

$$QuantumCircuit = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ $$QuantumCircuit^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Given the simplicity of the matrix representing the circuit, the following calculations can easily be done by hand.

$$f(0) = 1 \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$QuantumCircuit \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$f(1) = 0 \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$QuantumCircuit \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

These calculations demonstrate that the quantum circuit is a valid algoritm for the calculation of f(x). We now demonstrate parallel computation by putting |x> in a balanced superposition of |0> and |1>. As shown below, the operation of the circuit yields a superposition of the previous results. The function has been evaluated for both values of x in a single pass through the circuit.

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad QuantumCircuit \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.707 \\ 0.707 \\ 0 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \left[\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right] = \frac{1}{\sqrt{2}} \cdot \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$$

Haroche and Raimond describe this process as follows: "By superposing the inputs of a computation, one operates the machine 'in parallel', making it compute simultaneously all the values of a function and keeping its state, before any final bit detection is performed, suspended in a coherent superposition of all the possible outcomes." In summary, simple calculations have demonstrated how a quantum circuit can function as an algorithm for theevaluation of a mathematical function, and how the same circuit is capable of parallel evaluations of that function.

Input	Operation	Intermediate	Operation	Output
$\lvert 00 \rangle$		$\lvert 00 \rangle$		$\lvert 01 \rangle$
$\lvert 10 \rangle$	$\xrightarrow{CNOT}$	$\lvert 11 \rangle$	$\xrightarrow{I \otimes NOT}$	$\lvert 10 \rangle$

$$\frac{1}{\sqrt{2}} \left[\lvert 0 \rangle + \lvert 1 \rangle \right] \lvert 0 \rangle = \frac{1}{\sqrt{2}} \left[\lvert 00 \rangle + \lvert 10 \rangle \right] \qquad \frac{1}{\sqrt{2}} \left[\lvert 00 \rangle + \lvert 11 \rangle \right] \qquad \frac{1}{\sqrt{2}} \left[\lvert 01 \rangle + \lvert 10 \rangle \right]$$

However, as Haroche and Raimond note, on a practical level only one result can be realized for each operation of the circuit because on measurement the superposition created by the circuit collapses to one of the states forming the superposition. This is simulated with projection operators ($|0\rangle\langle0|$ and $|1\rangle\langle1|$) on both registers for the four possible measurement outcomes for each value of x.

$$f(0)=0 \qquad \left[kronecker\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}^T,\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}^T\right]\cdot QuantumCircuit\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\right]^2 = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$f(0)=1 \qquad \left[kronecker\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}^T,\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}^T\right]\cdot QuantumCircuit\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\right]^2 = \begin{pmatrix}0\\0.5\\0\\0\end{pmatrix}$$

$$f(1)=0 \qquad \left[kronecker\left[\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}^T,\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}^T\right]\cdot QuantumCircuit\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\right]^2 = \begin{pmatrix}0\\0\\0.5\\0\end{pmatrix}$$

$$f(1)=1 \qquad \left[kronecker\left[\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}^T,\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}^T\right]\cdot QuantumCircuit\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\right]^2 = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

As Haroche and Raimond write, "It is, however, one thing to compute potentially at once all the values of f(x) and quite another to be able to exploit this quantum parallelism and extract from it more information than from a mundane classical computation. The final stage of information acquisition must always be a measurement." Therefore, the exploitation of quantum parallelism for practical purposes such as searches and factorization requires more elaborate quantum circuits than the one presented here.
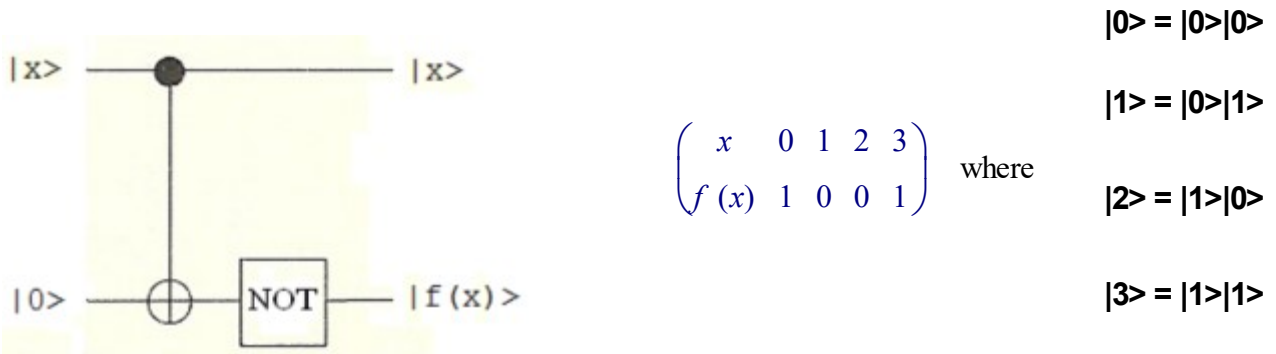
Truth tables for quantum circuit elements:

Identity $\begin{pmatrix} 0 & to & 0 \\ 1 & to & 1 \end{pmatrix}$ NOT $\begin{pmatrix} 0 & to & 1 \\ 1 & to & 0 \end{pmatrix}$ CNOT

Decimal	Binary	to	Binary	Decimal
0	00	to	00	0
1	01	to	01	1
2	10	to	11	3
3	11	to	10	2

X. An Illustration of the Deutsch-Jozsa Algorithm

Th D-J Algorithm demonstrates that there is a problem for which a QC runs faster than a Classic Computer.
Specifically, given a boolean function whose input is 1 bit   f: : {0,1} --> {0,1}, is it constant?
The following circuit produces the table of results to its right. The top wires carry the value of x and the circuit places f(x) on the bottom wire. As is shown in the previous Section (IX), this circuit can also operate in parallel accepting as input all x-values and returning on the bottom wire a superposition of all values of f(x).

$$\begin{pmatrix} x & 0 & 1 & 2 & 3 \\ f(x) & 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{where}$$

$$|0> = |0>|0>$$

$$|1> = |0>|1>$$

$$|2> = |1>|0>$$

$$|3> = |1>|1>$$

The function belongs to the balanced category because it produces 0 and 1 with equal frequency. A modification of this circuit (Deutsch-Jozsa algoritm, p. 298 in *The Quest for the Quantum Computer*, by Julian Brown) answers the question of whether the function is constant or balanced. Naturally we already know the answer, so this is a simple demonstration that the circuit works.

The input is **|0>|0>|1>** followed by a Hadamard gate on each wire, as shown in the circuit shown below. As is well known the Hadamard operation creates the following superposition states.

$$H \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad H \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Therefore the Hadamard operation transforms the input state to the following three-qubit state which is fed to the quantum circuit.

$$\frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2 \cdot \sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

The following matrices are required to execute the circuit. $I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $NOT := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $H := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$CNOT := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad CnNOT := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

After the portion of the quantum circuit shown above, Hadamard gates are added to the top two wires, as shown in the circuit on 2nd page down. The matrix representing the circuit is assembled using tensor matrix multiplication and then allowed to operate on the wave function. The full circuit is shown below.

$QCkt := kronecker(H, kronecker(H, I)) \cdot kronecker(I, kronecker(I, NOT)) \cdot kronecker(I, CNOT) \cdot CnNOT$

$$QCkt = \frac{1}{2} \cdot \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & -1 & 1 & 0 \end{pmatrix} \qquad QCkt \cdot \frac{1}{2 \cdot \sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.707 \\ 0.707 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Next the qubits on the top two wires are measured. If both are |0> the function is constant, but if at least one is |1> the function is balanced. The measurement on the top wires is implemented with projection operators |0><0| and |1><1|, and confirms that the function is not constant but belongs to the balanced category.

The first qubit is not |0>.
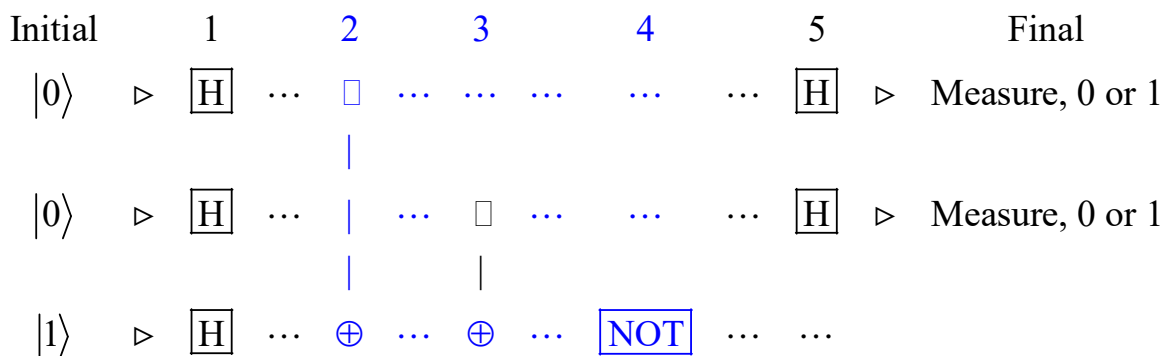$$kronecker\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{T}, kronecker(I, I) \right] \cdot QCkt \cdot \frac{1}{2 \cdot \sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The second qubit is not |0>. $kronecker\left[I, kronecker\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}^T, I\right]\right]\cdot QCkt\cdot\dfrac{1}{2\cdot\sqrt{2}}\cdot\begin{pmatrix}1\\-1\\1\\-1\\1\\-1\\1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}$

The first qubit is **|1>.** $kronecker\left[\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}^T, kronecker(I, I)\right]\cdot QCkt\cdot\dfrac{1}{2\cdot\sqrt{2}}\cdot\begin{pmatrix}1\\-1\\1\\-1\\1\\-1\\1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\\0\\-0.707\\0.707\end{pmatrix}$

The second qubit is **|1>.** $kronecker\left[I, kronecker\left[\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}^T, I\right]\right]\cdot QCkt\cdot\dfrac{1}{2\cdot\sqrt{2}}\cdot\begin{pmatrix}1\\-1\\1\\-1\\1\\-1\\1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\\0\\-0.707\\0.707\end{pmatrix}$

The following illustrates an algebraic analysis of the Deutsch-Jozsa algorithm.

	Initial	1	2	3	4	5	Final
$|0\rangle$	▷	H	⋯ ☐ ⋯	⋯ ⋯	⋯	⋯ H	▷ Measure, 0 or 1
$|0\rangle$	▷	H	⋯		⋯ ☐ ⋯	⋯	⋯ H	▷ Measure, 0 or 1
$|1\rangle$	▷	H	⋯ ⊕ ⋯	⊕ ⋯	NOT	⋯ ⋯	

$$H|0\rangle \rightarrow \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right] \qquad H|1\rangle \rightarrow \frac{1}{\sqrt{2}}\left[|0\rangle - |1\rangle\right]$$

NOT

$$\begin{pmatrix} 0 & ' & 1 \\ 1 & ' & 0 \end{pmatrix}$$

CNO

Decimal	Binary	'	Binary	Decimal
0	00	'	00	0
1	01	'	01	1
2	10	'	11	3
3	11	'	10	2

CnNO

Decimal	Binary	'	Binary	Decimal
0	000	'	000	0
1	001	'	001	1
2	010	'	010	2
3	011	'	011	3
4	100	'	101	5
5	101	'	100	4
6	110	'	111	7
7	111	'	110	6

$$|001\rangle$$

$$H \otimes H \otimes H$$

$$\frac{1}{\sqrt{2}}\big[|0\rangle + |1\rangle\big]\frac{1}{\sqrt{2}}\big[|0\rangle + |1\rangle\big]\frac{1}{\sqrt{2}}\big[|0\rangle - |1\rangle\big] = \frac{1}{2\sqrt{2}}\big[|000\rangle - |001\rangle + |010\rangle - |011\rangle + |100\rangle - |101\rangle + |110\rangle - |111\rangle\big]$$

$$CnNOT$$

$$\frac{1}{2\sqrt{2}}\big[|000\rangle - |001\rangle + |010\rangle - |011\rangle + |101\rangle - |100\rangle + |111\rangle - |110\rangle\big]$$

$$I \otimes CNOT$$

$$\frac{1}{2\sqrt{2}}\big[|000\rangle - |001\rangle + |011\rangle - |010\rangle + |101\rangle - |100\rangle + |110\rangle - |111\rangle\big]$$

$$I \otimes I \otimes NOT$$

$$\frac{1}{\sqrt{2}}\big[|0\rangle - |1\rangle\big]\frac{1}{\sqrt{2}}\big[|0\rangle - |1\rangle\big]\frac{1}{\sqrt{2}}\big[|1\rangle - |0\rangle\big]$$

$$H \otimes H \otimes I$$

$$|1\rangle|1\rangle\frac{1}{\sqrt{2}}\big(|1\rangle - |0\rangle\big)$$

Since the top wires contain $|1>$, the function is balanced. **This algorithm illustrates the roles of superposition, entanglement and interference in quantum computation**. Regarding the latter, it is destructive interference in the last step that eliminates unwanted outcomes yielding the final result on the last line.

One pass through the quantum circuit answers the question (is the function balanced or constrant) that would take 4 calculations on a classical computer. Thus given this problem, a QC is faster than a classical computer.

The interference that occurs in the last step is illustrated by letting $|a> = |0>$ and $|b> = |1>$ and carrying out Hadamard transforms on the first two qubits.

$$\frac{1}{4\cdot\sqrt{2}}\cdot\begin{bmatrix} (a_1 + b_1)\cdot(a_2 + b_2)\cdot b_3 - (a_1 + b_1)\cdot(a_2 + b_2)\cdot a_3 \ldots \\ + (a_1 + b_1)\cdot(a_2 - b_2)\cdot b_3 - (a_1 + b_1)\cdot(a_2 - b_2)\cdot a_3 \ldots \\ + (a_1 - b_1)\cdot(a_2 + b_2)\cdot b_3 - (a_1 - b_1)\cdot(a_2 + b_2)\cdot a_3 \ldots \\ + (a_1 - b_1)\cdot(a_2 - b_2)\cdot b_3 - (a_1 - b_1)\cdot(a_2 - b_2)\cdot a_3 \end{bmatrix} \; simplify \rightarrow -\frac{\sqrt{2}\cdot a_1\cdot a_2\cdot(a_3 - b_3)}{2}$$

XI. Quantum Mechanical Calculations Illuminated with Dirac Notation

The following example is from "*PHYSICAL CHEMISTRY for the Chemical Sciences*", Raymond Chang"

A Particle in a Box in a One Dimensional Box

At the end of the nineteenth century, there were new experimental results that could not be explained by the so-called classical theories of physics. In 1900, the German physicist Max Planck proposed the quantum theory to explain one of these experiments. In this chapter, we take a historical approach and follow the early development of quantum theory.

Consider a particle of mass m confined to a one-dimensional box of length L. We again assume that the particle has zero potential energy inside the box ^ or on the line segment h ; that is, V = 0. The particle has only kinetic energy. At each end of the box is a wall of infinite potential energy, so there is no probability of finding the particle at the walls or outside the box. For simplicity, we chose the line segment to start at the origin, so x is restricted by $0 \leq x \leq L$, See Figure below. The Schrödinger equation is similar to that for the free particle, with the difference being that the value of x is constrained by the size of the box.
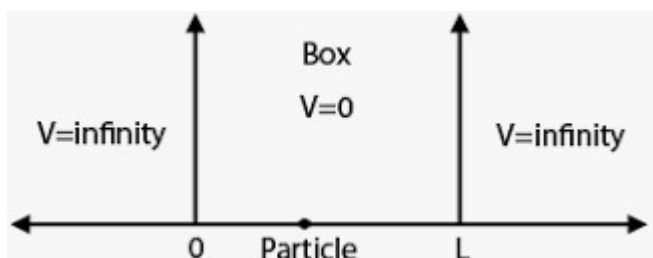
$$\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x) \qquad 0 \leq x \leq L$$

For the particle-in-a-box Schrödinger equation, let us try a trial wave function,

$$\psi(x) = A\,\sin(kx) + B\,\cos(kx)$$

The particle-in-a-box (PIB) problem is exactly soluble and the solution is calculated below for the first 20 eigenstates. All calculations will be carried out in atomic units (h = 2π) for a particle of unit mass in a 1 D box.

$$n := 1..20 \qquad \Psi(n,x) := \sqrt{2}\cdot\sin(n\cdot\pi\cdot x) \qquad E_n := \frac{n^2\cdot\pi^2}{2}$$

The first five energy eigenvalues are:

$E_1 = 4.935 \qquad E_2 = 19.739 \qquad E_3 = 44.413 \qquad E_4 = 78.957 \qquad E_5 = 123.37$

The first three eigenfunctions are displayed below. $\qquad x := 0,.02..1$

$\Psi(1,x)$
$\Psi(2,x)$
$\Psi(3,x)$
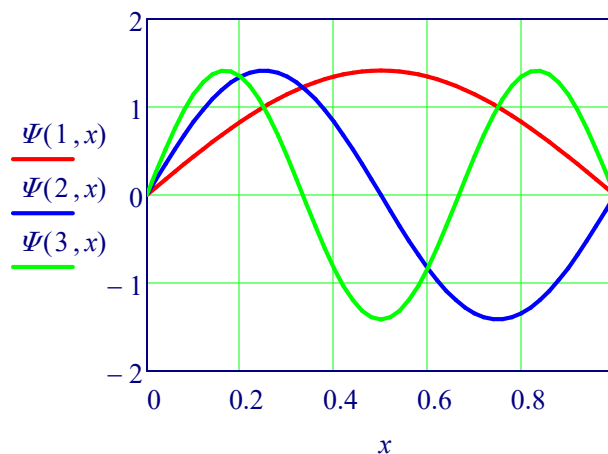
The PIB eigenfunctions form a complete basis set, and therefore other functions can be written as linear combinations in this basis set. For example, Φ, χ, and Γ are three trial functions that satisfy the boundary conditions for the particle in a 1 bohr box.

$$\Phi(x) := \sqrt{30}\cdot\left(x - x^2\right) \qquad \chi(x) := \sqrt{105}\cdot\left(x^2 - x^3\right) \qquad \Gamma(x) := \sqrt{105}\cdot x\cdot(1 - x)^2$$

In Dirac bracket notation we can express each of these functions as a linear combination in the basis set. For example, for Φ we have,

$$\langle x|\Phi\rangle = \sum_n \langle x|\Psi_n\rangle\langle\Psi_n|\Phi\rangle = \sum_n \langle x|\Psi_n\rangle\int_0^1 \langle\Psi_n|x\rangle\langle x|\Phi\rangle dx = \sum_n \Psi_n(x)\int_0^1 \Psi_n^*(x)\Phi(x)dx = \sum_n \Psi_n(x)a_n$$

Here both the finite and continuous completeness relations have been used:

$$\sum_n |\Psi_n\rangle\langle\Psi_n| = 1 \quad \text{and} \quad \int |x\rangle\langle x|dx = 1$$

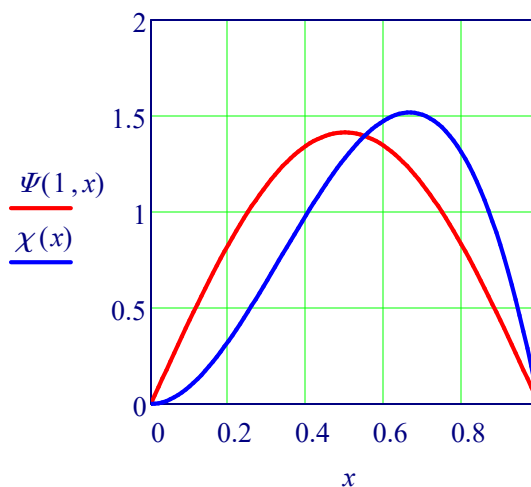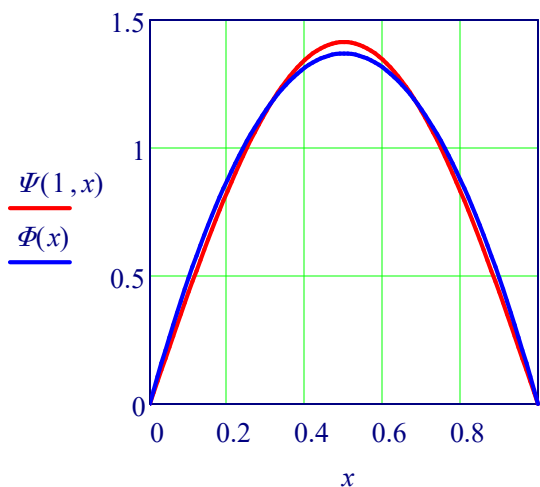The various overlap integrals for the three trial function (a_n, b_n, and c_n) are evaluated below.

$$a_n := \int_0^1 \Psi(n,x)\cdot\Phi(x)\, dx \qquad b_n := \int_0^1 \Psi(n,x)\cdot\chi(x)\, dx \qquad c_n := \int_0^1 \Psi(n,x)\cdot\Gamma(x)\, dx$$
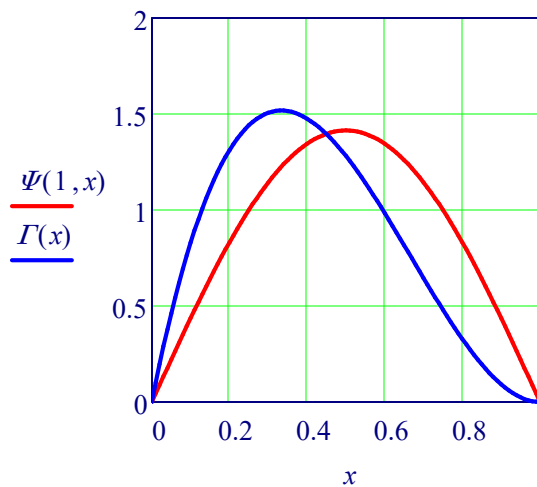
As shown above, these overlap integrals are set up as follows:

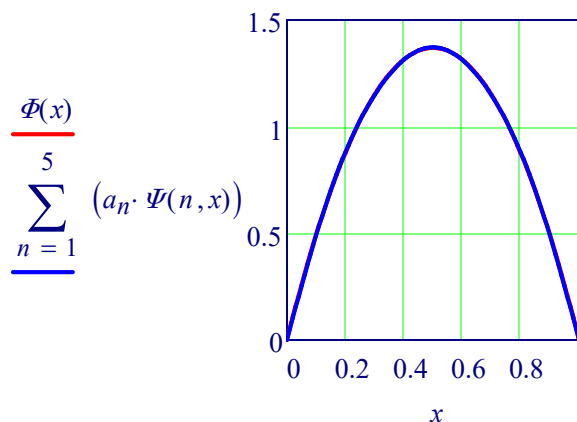$$a_n = \langle\Psi_n|\Phi\rangle = \int_0^1 \langle\Psi_n|x\rangle\langle x|\Phi\rangle dx = \int_0^1 \Psi_n^*(x)\Phi(x)dx$$

The figures shown below demonstrate that only Φ is a reasonable representative for the ground state wavefunction.

$$x := 0, .01 .. 1$$

$\Psi(1,x)$

$\Gamma(x)$

However, if Φ is written as a linear combination of the first 5 PIB eigenfunctions, one gets two functions that are essentially indistinquishable from one another.

$\Phi(x)$

$$\sum_{n=1}^{5} \left(a_n \cdot \Psi(n,x) \right)$$

The same, of course, is true for χ and Γ, as is demonstrated in the graphs shown below.

$\chi(x)$

$$\sum_{n=1}^{5} \left(b_n \cdot \Psi(n,x) \right)$$

$\Gamma(x)$

$$\sum_{n=1}^{5} \left(c_n \cdot \Psi(n,x) \right)$$
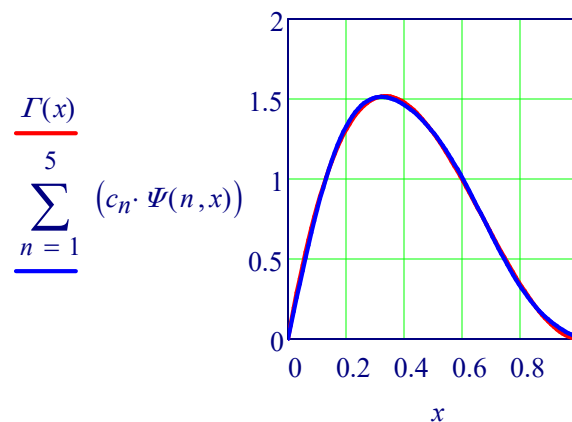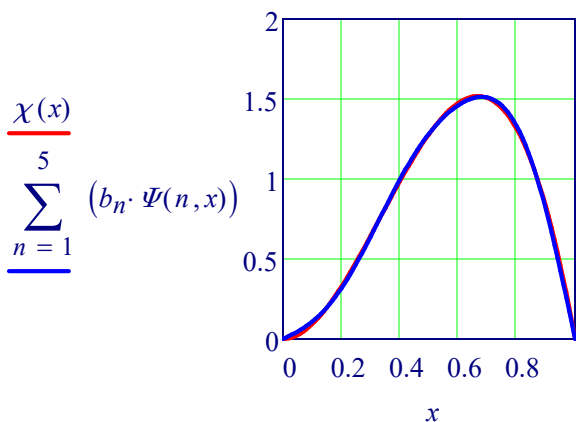
Traditionally we use energy as a criterion for the quality of a trial wavefunction by evaluating the variational integral in the following way. $\Phi(x)$ is the best trial function because it gives the lowest energy.

$$\int_0^1 \Phi(x) \cdot -\frac{1}{2} \cdot \frac{d^2}{dx^2} \Phi(x)\, dx = 5 \qquad \int_0^1 \chi(x) \cdot -\frac{1}{2} \cdot \frac{d^2}{dx^2} \chi(x)\, dx = 7 \qquad \int_0^1 \Gamma(x) \cdot -\frac{1}{2} \cdot \frac{d^2}{dx^2} \Gamma(x)\, dx = 7$$

With Dirac notation we would write:

$$\langle E \rangle = \langle \Phi | \hat{H} | \Phi \rangle = \sum_n \langle \Phi | \hat{H} | \Psi_n \rangle \langle \Psi_n | \Phi \rangle = \sum_n \langle \Phi | \Psi_n \rangle E_n \langle \Psi_n | \Phi \rangle = \sum_n a_n^2 E_n$$

Thus we easily show the same result.

$$\sum_n \left[(a_n)^2 \cdot E_n \right] = 5 \qquad \sum_n \left[(b_n)^2 \cdot E_n \right] = 6.999 \qquad \sum_n \left[(c_n)^2 \cdot E_n \right] = 6.999$$

We now show, belatedly, that the three trial functions are normalized by both methods.

$$\int_0^1 \Phi(x)^2\, dx = 1 \qquad \int_0^1 \chi(x)^2\, dx = 1 \qquad \int_0^1 \Gamma(x)^2\, dx = 1$$

In Dirac bracket notation this is written as:
$$\langle \Phi | \Phi \rangle = \sum_n \langle \Phi | \Psi_n \rangle \langle \Psi_n | \Phi \rangle = \sum_n a_n^2$$

$$\sum_n (a_n)^2 = 1 \qquad \sum_n (b_n)^2 = 1 \qquad \sum_n (c_n)^2 = 1$$

We now calculate some over-lap integrals:

$$\int_0^1 \Phi(x) \cdot \chi(x)\, dx = 0.935 \qquad \int_0^1 \Phi(x) \cdot \Gamma(x)\, dx = 0.935 \qquad \int_0^1 \chi(x) \cdot \Gamma(x)\, dx = 0.75$$

In Dirac notation this is formulated as:
$$\langle \Phi | \Gamma \rangle = \sum_n \langle \Phi | \Psi_n \rangle \langle \Psi_n | \Gamma \rangle = \sum_n a_n c_n$$

$$\sum_n (a_n \cdot b_n) = 0.935 \qquad \sum_n (a_n \cdot c_n) = 0.935 \qquad \sum_n (b_n \cdot c_n) = 0.75$$

As a final exercise we calculate the expectation value for position using the three trial wave functions. In bracket notation this calculation is set up most directly as follows.

$$\langle \Phi | \hat{x} | \Phi \rangle = \int_0^1 \langle \Phi | \hat{x} | x \rangle \langle x | \Phi \rangle\, dx = \int_0^1 \langle \Phi | x \rangle x \langle x | \Phi \rangle\, dx = \int_0^1 \Phi(x)^* x \Phi(x)\, dx$$

where we have employed the eigenvalue equation for the position operator: $\qquad \hat{x} | x \rangle = | x \rangle x$

Evaluation of the integral on the right for each trial function is shown below. Naturally the results are consistent with the shapes of the trial wave functions shown in the first figure.

$$\int_0^1 \Phi(x) \cdot x \cdot \Phi(x) \, dx = 0.5 \qquad \int_0^1 \chi(x) \cdot x \cdot \chi(x) \, dx = 0.625 \qquad \int_0^1 \Gamma(x) \cdot x \cdot \Gamma(x) \, dx = 0.375$$

Although it is computationally less expedient, it is instructive to expand these calculations in terms of the PIB eigenfunctions.

$$\langle \Phi | \hat{x} | \Phi \rangle = \sum_m \sum_n \int_0^1 \langle \Phi | \Psi_m \rangle \langle \Psi_m | x \rangle x \langle x | \Psi_n \rangle \langle \Psi_n | \Phi \rangle \, dx = \sum_m \sum_n a_m^* a_n \int_0^1 \sqrt{2} \sin(m\pi x) x \sqrt{2} \sin(n\pi x) \, dx$$

Truncating the calculation after five PIB eigenfunctions yields the same results as obtained with the integrals above.

$$\sum_{m=1}^5 \sum_{n=1}^5 \left(a_m \cdot a_n \cdot \int_0^1 \sqrt{2} \cdot \sin(m \cdot \pi \cdot x) \cdot x \cdot \sqrt{2} \cdot \sin(n \cdot \pi \cdot x) \, dx \right) = 0.5$$

$$\sum_{m=1}^5 \sum_{n=1}^5 \left(b_m \cdot b_n \cdot \int_0^1 \sqrt{2} \cdot \sin(m \cdot \pi \cdot x) \cdot x \cdot \sqrt{2} \cdot \sin(n \cdot \pi \cdot x) \, dx \right) = 0.625$$

$$\sum_{m=1}^5 \sum_{n=1}^5 \left(c_m \cdot c_n \cdot \int_0^1 \sqrt{2} \cdot \sin(m \cdot \pi \cdot x) \cdot x \cdot \sqrt{2} \cdot \sin(n \cdot \pi \cdot x) \, dx \right) = 0.375$$

XII. Quantum Restrictions on Cloning

Reference: Frank Rioux
Emeritus Professor of Chemistry
CSB|SJU

Suppose a quantum copier exists which is able to carry out the following cloning operation.

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{\text{Clone}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Next the cloning operation (using the same copier) is carried out on the general qubit shown below.

$$\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \xrightarrow{\text{Clone}} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \otimes \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} = \begin{pmatrix} \cos^2(\theta) \\ \cos(\theta)\sin(\theta) \\ \sin(\theta)\cos(\theta) \\ \sin^2(\theta) \end{pmatrix}$$

Quantum transformations are unitary, meaning probability is preserved. This requires that the scalar products of the initial and final states must be the same.

Initial state:
$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sin(\theta)$$

Final state:
$$\begin{pmatrix} \cos^2(\theta) & \cos(\theta)\sin(\theta) & \sin(\theta)\cos(\theta) & \sin^2(\theta) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \sin^2(\theta)$$

It is clear from this analysis that quantum theory puts a significant restriction on copying. Only states for which $\sin(\theta) = 0$ or 1 (0 and 90 degrees) can be copied by the original cloner.

In conclusion, two quotes from Wootters and Zurek, *Physics Today*, February 2009, page 76.

> Perfect copying can be achieved only when the two states are orthogonal, and even then one can copy those two states (...) only with a copier specifically built for that set of states.

> In sum, one cannot make a perfect copy of an unknown quantum state, since, without prior knowledge, it is impossible to select the right copier for the job. That formulation is one common way of stating the no-cloning theorem.

An equivalent way to look at this (see arXiv:1701.00989v1) is to assume that a cloner exists for the V-H polarization states.

$$\hat{C}|V\rangle|X\rangle = |V\rangle|V\rangle \qquad \hat{C}|H\rangle|X\rangle = |H\rangle|H\rangle$$

A diagonally polarized photon is a superposition of the V-H polarization states.

$$|D\rangle = \frac{1}{\sqrt{2}}(|V\rangle + |H\rangle)$$

However, due to the linearity of quantum mechanics **the V-H cloner cannot clone a diagonally polarized photon.**

$$\hat{C}|D\rangle|X\rangle = \hat{C}\frac{1}{\sqrt{2}}(|V\rangle + |H\rangle)|X\rangle = \frac{1}{\sqrt{2}}\left(\hat{C}|V\rangle|X\rangle + \hat{C}|H\rangle|X\rangle\right) = \frac{1}{\sqrt{2}}(|V\rangle|V\rangle + |H\rangle|H\rangle)$$

$$\hat{C}|D\rangle|X\rangle \neq |D\rangle|D\rangle = \frac{1}{2}(|V\rangle|V\rangle + |V\rangle|H\rangle + |H\rangle|V\rangle + |H\rangle|H\rangle)$$

XIII. Factoring Using Shor's Quantum Algorithm

Reference: Frank Rioux
Emeritus Professor of Chemistry
CSB|SJU

This tutorial presents a toy calculation dealing with the quantum factorization of 15 using Shor's algorithm. The first step is to find the period of a^x **modulo 15**, where **a** is chosen randomly.

$$a := 4 \qquad N := 15 \quad f(x) := mod(a^x, N) \quad Q := 4 \quad x := 0 .. Q - 1 \quad x = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} \quad f(x) = \begin{pmatrix} 1 \\ 4 \\ 1 \\ 4 \end{pmatrix}$$

We proceed by ignoring the fact that we can see by inspection that the period of f(x) is 2 and demonstrate how it is determined using a quantum (discrete) Fourier transform. After the registers are loaded with x and f(x) using a **quantum** computer, they exist in the following **superposition**.

$$\frac{1}{\sqrt{Q}} \sum_{x=0}^{Q-1} |x\rangle |f(x)\rangle = \frac{1}{2} \left[|0\rangle |1\rangle + |1\rangle |4\rangle + |2\rangle |1\rangle + |3\rangle |4\rangle + \cdots \right]$$

The next step is to find the period of f(x) by performing a quantum Fourier transform (QFT) on the input register $|\mathbf{x}\rangle$.

$$Q := 4 \quad mm := 0 .. Q - 1 \quad n := 0 .. Q - 1 \quad QFT_{mm,n} := \frac{1}{\sqrt{Q}} \cdot exp\left(i \cdot \frac{2 \cdot \pi \cdot mm \cdot n}{Q} \right)$$

$$x = 0 \qquad QFT \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{pmatrix} \qquad\qquad x = 1 \qquad QFT \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5i \\ -0.5 \\ -0.5i \end{pmatrix}$$

$$x = 2 \quad QFT \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 \\ -0.5 \\ 0.5 \\ -0.5 \end{pmatrix} \qquad\qquad x = 3 \quad QFT \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.5 \\ -0.5i \\ -0.5 \\ 0.5i \end{pmatrix}$$

The operation of the QFT on the x-register is expressed algebraically in the middle term below. Quantum interference in this term yields the result on the right which shows a period of 2 on the x-register.

$$\frac{1}{4}\big[|0\rangle + |1\rangle + |2\rangle + |3\rangle\big]|1\rangle$$

$$QFT(x)\frac{1}{2}\big[|0\rangle|1\rangle + |1\rangle|4\rangle + |2\rangle|1\rangle + |3\rangle|4\rangle\big] \quad = \quad +\frac{1}{4}\big[|0\rangle + i|1\rangle - |2\rangle - i|3\rangle\big]|4\rangle \quad = \quad \frac{1}{2}\big[|0\rangle\big(|1\rangle + |4\rangle\big) + |2\rangle\big(|1\rangle - |4\rangle\big)\big]$$

$$+\frac{1}{4}\big[|0\rangle - |1\rangle + |2\rangle - |3\rangle\big]|1\rangle$$

$$+\frac{1}{4}\big[|0\rangle - i|1\rangle - |2\rangle + i|3\rangle\big]|4\rangle$$

The next step is to use the Euclidian algorithm by calculating the greatest common divisor of two functions involving the period and **a**, and the number to be factored, **N**. This yields the prime factors of 15.

$$period := 2 \qquad gcd\left(a^{\frac{period}{2}} - 1, N\right) = 3 \qquad gcd\left(a^{\frac{period}{2}} + 1, N\right) = 5$$

Figure 5 in "Quantum Computation," by David P. DiVincenzo, *Science* **270**, 258 (1995) provides a succinct graphical illustration of the steps of Shor's factorization algorithm.
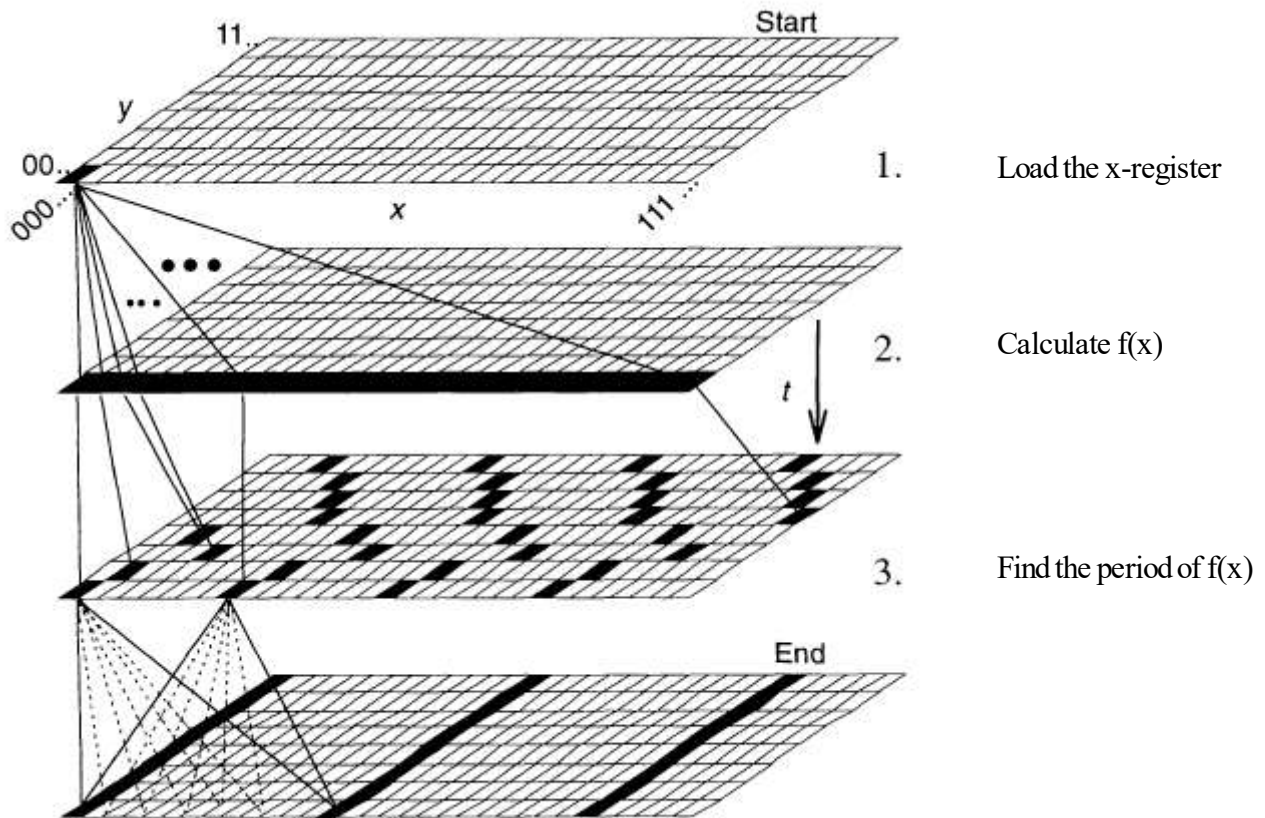
XIV. An Entanglement Swapping Protocol

Reference: Frank Rioux
Emeritus Professor of Chemistry
CSB|SJU

In the field of quantum information interference, superpositions and entangled states are essential resources. Entanglement, a non-factorable superposition, is routinely achieved when two photons are emitted from the same source, say a parametric down converter (PDC). Entanglement swapping involves the transfer of entanglement to two photons that were produced independently and never previously interacted. The Bell states are the four maximally entangled two-qubit entangled basis for a four-dimensional Hilbert space and play an essential role in quantum information theory and technology, including teleportation and entanglement swapping. The Bell states are shown below.

$$\Phi_p = \frac{1}{\sqrt{2}} \cdot \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \quad \Phi_p := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \Phi_m = \frac{1}{\sqrt{2}} \cdot \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \quad \Phi_m := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\Psi_p = \frac{1}{\sqrt{2}} \cdot \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \quad \Psi_p := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \qquad \Psi_m = \frac{1}{\sqrt{2}} \cdot \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \quad \Psi_m := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

A four-qubit state is prepared in which photons 1 and 2 are entangled in Bell state Φ_p, and photons 3 and 4 are entangled in Bell state Ψ_m. The state multiplication below is understood to be tensor vector multiplication.

$$\Psi = \Phi_p \cdot \Psi_m = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \quad \Psi := \frac{1}{2} \cdot (0\ 1\ -1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ -1\ 0)^T \quad I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Four Bell state measurements are now made on photons 2 and 3 which entangles photons 1 and 4.
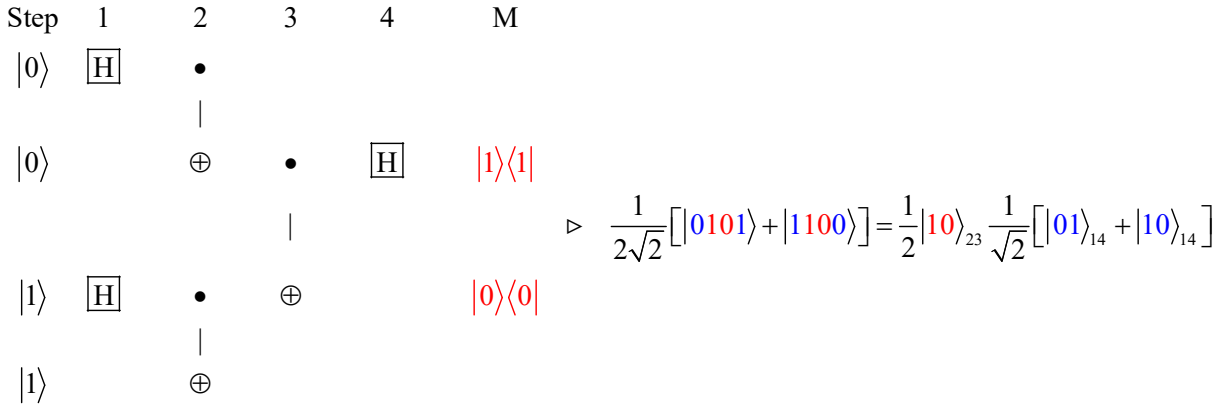Projection of photons 2 and 3 onto Φ_p projects photons 1 and 4 onto Ψ_m.

$$\left(kronecker\left(I, kronecker\left(\Phi_p \cdot \Phi_p^T, I \right) \right) \cdot \Psi \right)^T = (0\ \ 0.25\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0.25\ \ -0.25\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ -0.25\ \ 0)$$

$$\frac{1}{2\cdot\sqrt{2}} \cdot \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]^T = \frac{1}{4} \cdot (0\ 1\ 0\ 0\ 0\ 0\ 1\ -1\ 0\ 0\ 0\ 0\ 0\ -1\ 0)$$

Projection of photons 2 and 3 onto Φ_m projects photons 1 and 4 onto Ψ_p.

$$\left(kronecker\left(I,kronecker\left(\Phi_m \cdot \Phi_m^T, I\right)\right)\cdot \Psi\right)^T = (0 \quad 0.25 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -0.25 \quad 0.25 \quad 0 \quad 0 \quad 0 \quad 0 \quad -0.25 \quad 0)$$

$$\frac{1}{2\cdot\sqrt{2}}\cdot\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\right]^T = \frac{1}{4}\cdot(0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -1 \quad 0)$$

Here's a quantum circuit that accomplishes this entanglement swap.

Step	1	2	3	4	M
$|0\rangle$	H	•			
		│			
$|0\rangle$		⊕	•	H	$|1\rangle\langle1|$
			│		
$|1\rangle$	H	•	⊕		$|0\rangle\langle0|$
		│			
$|1\rangle$		⊕			

$\triangleright \quad \frac{1}{2\sqrt{2}}\left[|0101\rangle+|1100\rangle\right]=\frac{1}{2}|10\rangle_{23}\frac{1}{\sqrt{2}}\left[|01\rangle_{14}+|10\rangle_{14}\right]$

Projection of photons 2 and 3 onto Ψ_p projects photons 1 and 4 onto $-\Phi_m$.

$$\left(kronecker\left(I,kronecker\left(\Psi_p \cdot \Psi_p^T, I\right)\right)\cdot \Psi\right)^T = (0 \quad 0 \quad -0.25 \quad 0 \quad -0.25 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.25 \quad 0 \quad 0.25 \quad 0 \quad 0)$$

$$\frac{1}{2\cdot\sqrt{2}}\cdot\left[\begin{pmatrix}0\\1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}-\begin{pmatrix}1\\0\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\right]^T = \frac{1}{4}\cdot(0 \quad 0 \quad -1 \quad 0 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0)$$

Finally, projection of photons 2 and 3 onto Ψ_m projects photons 1 and 4 onto Φ_p.

$$\left(kronecker\left(I,kronecker\left(\Psi_m \cdot \Psi_m^T, I\right)\right)\cdot \Psi\right)^T = (0 \quad 0 \quad -0.25 \quad 0 \quad 0.25 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -0.25 \quad 0 \quad 0.25 \quad 0 \quad 0)$$

$$\frac{-1}{2\cdot\sqrt{2}}\cdot\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\right]^T = \frac{1}{4}\cdot(0 \quad 0 \quad -1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -1 \quad 0 \quad 1 \quad 0 \quad 0)$$

XV. QM Example: Simulating the Aharonov–Bohm Effect

The Aharonov–Bohm effect is a quantum-mechanical phenomenon in which an electrically charged particle is influenced by the vector potential **A** in regions in which the magnetic field **B** is zero. A beam of monoenergetic electrons passes through a double slit on opposite sides of a solenoid. In QM, the same particle can travel between two paths. The expected interference pattern of the waves going through the two slits is shifted by an additional phase difference ϕ when the solenoid encloses a magnetic field, despite the magnetic field being zero in the regions through which the electrons pass. This can be observed experimentally by the horizontal displacement of the interference fringes.
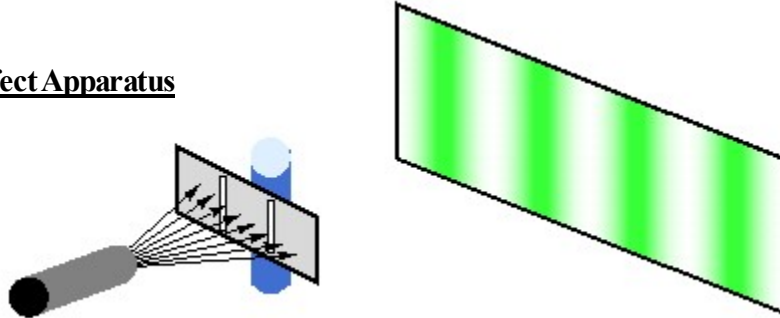
Aharonov–Bohm Effect Apparatus

Illustration of how double-slit experiment in which Aharonov–Bohm effect can be observed: electrons pass through two slits, interfering at an observation screen, with the interference pattern shifted when a magnetic field B is turned on in the cylindrical solenoid. **The effect on the interference fringes is calculated and displayed below.**

Slit Positions	Slit Width	AB Relative Phase Shift
$x_L := 1 \quad x_R := 2$	$\delta := 0.2$	Phase $\phi := \pi$ for $exp(i \cdot \phi)$

$$\Psi(p) := \frac{1}{\sqrt{2}} \left(\int_{x_L - \frac{\delta}{2}}^{x_L + \frac{\delta}{2}} \frac{1}{\sqrt{2 \cdot \pi}} \cdot exp(-i \cdot p \cdot x) \cdot \frac{1}{\sqrt{\delta}} \, dx + \int_{x_R - \frac{\delta}{2}}^{x_R + \frac{\delta}{2}} \frac{1}{\sqrt{2 \cdot \pi}} \cdot exp(-i \cdot p \cdot x) \cdot \frac{1}{\sqrt{\delta}} \, dx \right)$$

$$\Phi(p, \phi) := \frac{1}{\sqrt{2}} \left(\int_{x_L - \frac{\delta}{2}}^{x_L + \frac{\delta}{2}} \frac{1}{\sqrt{2 \cdot \pi}} \cdot exp(-i \cdot p \cdot x) \cdot \frac{1}{\sqrt{\delta}} \, dx + exp(i \cdot \phi) \cdot \int_{x_R - \frac{\delta}{2}}^{x_R + \frac{\delta}{2}} \frac{1}{\sqrt{2 \cdot \pi}} \cdot exp(-i \cdot p \cdot x) \cdot \frac{1}{\sqrt{\delta}} \, dx \right)$$

Aharonov–Bohm Effect Interference Pattern with 180 degree Phase Difference

$(|\Psi(p)|)^2$

$(|\Phi(p, \pi)|)^2$
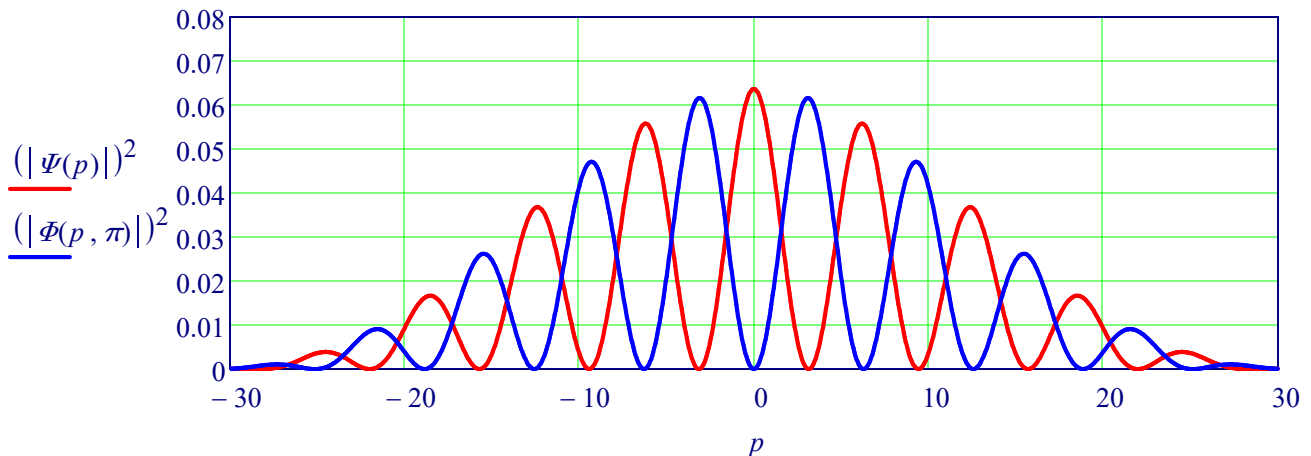
XVI. Solution of Schrödinger Wave Equation for Propogation of Electron

Given an electron of mass, m_e, velocity, v_e, kinetic energy of 1 eV

By Quantum Mechanics. it has an associated de Broglie wavelength, λ_e, and wavenumber k_0

Planck's Constant: $h := 6.626 \cdot 10^{-34} J \cdot s$

Given: $m_e := 9.10938 \cdot 10^{-31} kg$ $\quad v := 1 \cdot 10^3 \dfrac{m}{s}$ $\quad T := 1eV$ $\quad \lambda_e := \dfrac{h}{\sqrt{2 \cdot m_e \cdot T}} = 12.265 \cdot A$

Consider a monochromatic E Field plane wave associated with an electron which propagates in an isotropic and homogeneous medium: $E(r,t) = E_0 \cdot e^{[i \cdot (k \cdot r - \omega \cdot t)]}$

Associated with this electron is a wavenumber, k_e, amplitude, A

$$k_e := 8.637 \cdot 10^6 \cdot \dfrac{1}{m} \qquad A := \dfrac{1}{\sqrt{a \cdot \sqrt{\pi}}} \qquad h_{bar} := \dfrac{h}{2 \cdot \pi} \qquad \boxed{A_m := A \cdot \sqrt{m}}$$

$eV := 1.602 \cdot 10^{-19} J$

The electron has Kinetic Energy: $\quad E := \dfrac{h_{bar}^2 \cdot k_e^2}{2.m_e}$ $\quad E = 4.554 \times 10^{-25} J$ $\quad$ **Electron Frequency:** $\quad \omega = 2\pi \cdot f$

Equation of Traveling Wave: $\quad \boxed{\psi(x) = A \cdot e^{i \cdot (k \cdot x - \omega \cdot t)} + B \cdot e^{-i \cdot (k \cdot x + \omega \cdot t)}}$

To solve the one-dimensional Schrödinger equation for a free particle of mass m moving with velocity v, we can proceed a follows:

Solve Schrödinger's Wave Equation for the Quantum Wavefunction, Ψ(x, t)

$$\boxed{\dfrac{-\hbar^2}{2 \cdot m_e} \cdot \dfrac{\partial^2}{\partial x^2} \Psi = i \cdot \hbar \cdot \dfrac{\partial}{\partial t} \Psi} \qquad \Psi(x,t) = \int_{-\infty}^{\infty} \psi(k,x,t) \, dk$$

$$\boxed{\Psi(x,t) = \dfrac{A \cdot a}{\sqrt{2 \cdot \pi}} \cdot \int_{-\infty}^{\infty} \exp\left[\dfrac{-1}{2} \cdot a^2 \cdot (k - k_0)^2 + i \cdot k \cdot x - \dfrac{i \cdot \hbar \cdot t}{2 \cdot m_e} \cdot k^2\right] dk}$$
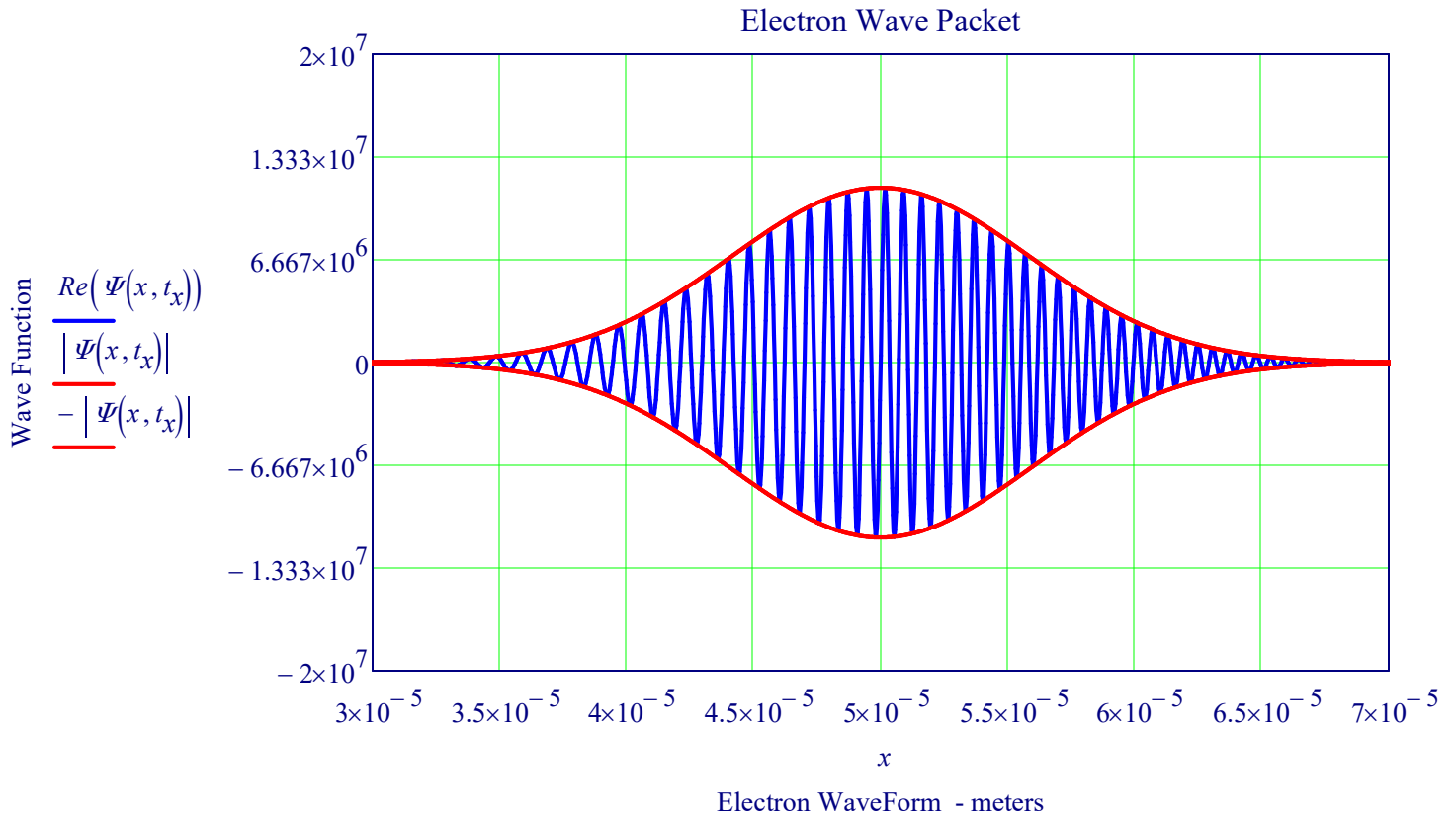
Evalute the Wavefunction over the Space and Time Region: $\quad a := 1 \cdot \mu m \qquad t_x := 50ns$

Solution for Ψ(x,t): $\quad \Psi(x,t) := \dfrac{A_m}{\sqrt{1 + \dfrac{i \cdot h_{bar} \cdot t}{m_e \cdot a^2}}} \cdot \exp\left[\dfrac{-\left(x^2 - 2 \cdot i \cdot a^2 \cdot k_e \cdot x + \dfrac{i \cdot h_{bar} \cdot t}{2.m_e} \cdot k_e^2 \cdot a^2\right)}{2 \cdot a^2 \cdot \left(1 + \dfrac{i \cdot h_{bar} \cdot t}{m_e \cdot a^2}\right)}\right]$

Plot Wavefunction Ψ(x,t) over Distance Range, x

Distance Range: $x := 10^{-5} \cdot 2m, \, 10^{-5} \cdot 2m + \left(\dfrac{10^{-5} \cdot 8 \cdot m - 10^{-5} \cdot 2m}{2000} \right) .. \, 10^{-5} \cdot 8m$

Electron Wave Packet

Wave Function: $Re\left(\Psi(x, t_x) \right)$, $\left| \Psi(x, t_x) \right|$, $-\left| \Psi(x, t_x) \right|$

x

Electron WaveForm - meters

Bibliography:

The Meaning of Quantum Theory, Jim Baggott, Chapter 4
Methodology using matrix and tensor algebra, Frank Rioux
Exploring the Quantum, Haroche and Raimond
The Quest for the Quantum Computer, by Julian Brown, page 298
Qubit Quantum Mechanics, Enrique Galvez, AJP 78, 510-519 (2010)
Quantum Computation, by David P. DiVincenzo, *Science* **270**, 258 (1995)
QM general formalism, Edited by F. M., page 36
A Course in Quantum Computing, Vol_1_Loceff, Foothill College, Creative Commons pg. 139